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HEISENBERG'S UNCERTAINTY RELATION MAY BE VIOLATED IN A SINGLE MEASUREMENT

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Abstract

The well-known Heisenberg's uncertainty relation is an inequality between uncertainties of canonically conjugate observables in a given state. In this interpretation, the Heisenberg's uncertainty relation is a rigorous mathematical theorem and is, therefore, always valid. However, the same inequality is often applied in the situation of measurement, where it is illustrated in a quite different way. The uncertainty relation is then an inequality connecting the precision (resolution) of the measurement of one observable and the uncertainty of the conjugate observable in the state arising after the measurement. It turns out that in such an interpretation the Heisenberg's inequality may be violated for some measurement readouts that emerge with small but finite probabilities. Making use of the uncertainties averaged in a special way over all possible measurement readouts, one may formulate an inequality of the type of Heisenberg's inequality but valid for any measurement.

Keywords: uncertainty relation, quantum measurement.

1. Introduction

The Heisenberg's uncertainty relations [1]

$$\Delta p \Delta q \geq \frac{\hbar}{2} \tag{1}$$

are (and always have been) one of the most widely discussed aspects of quantum mechanics. They are often mentioned in connection with measurements [2–8] However, the Heisenberg's uncertainty relations for a given state are derived in many handbooks quite rigorously while their applications to measurements are discussed on a qualitative level or only illustrated by examples. In the literature on the mathematically rigorous theory of quantum measurements, various types of uncertainty relations are considered in a general and precise way [9–11] but the physical analysis of the considered phenomena (especially measurements) is often insufficient.

Attentive analysis shows that there are two qualitatively different situations in which the uncertainty relation may be applied. The first situation arises if the uncertainty relation is an inequality for two observables in an (arbitrary) given state of the physical system. Then Δp, Δq in (1) are uncertainties (of the momentum and coordinate) in the given state ψ. Inequality (1) is always valid in this situation. The second situation is the situation of measurement. In this case one of the observables is measured while the second one obtains uncertainty as a result of the unavoidable influence of the measurement on

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the system. Inequality (1) is then interpreted in a quite different way — if the coordinate is measured with the precision Δq, then the momentum acquires the uncertainty Δp. It is commonly believed that the thus interpreted entities Δq, Δp also satisfy Eq. (1). Is this right?

The last statement is commonly believed to be valid but it is really, in a sense, approximately valid. However, it is not valid in a rigorous mathematical sense. This has been shown in [1] where the other Heisenberg-type inequality has been derived which is universally valid in the situation of measurement. Below these results will be briefly discussed. The uncertainty relations for measurements has also been investigated in [12, 13] within the framework of different approaches.

The question about uncertainty relations in measurements again became crucial after the long interesting discussion in the literature [8, 14–20]. It was suggested in the course of this discussion that the uncertainty relation is completely wrong in interferometric experiments with atoms [20]. However, this radical statement proved to be wrong [17,18] (see [1] for the detailed analysis of the mentioned discussion and especially of the experiment [20]).

2. Measurement-Induced Uncertainty Relation

The Heisenberg's uncertainty relations for the coordinate and linear momentum have the form (1), if the uncertainty of an observable A is the square average deflection of this observable from its mean value

$$\Delta A^2 = \langle\psi|(A - \langle\psi|A|\psi\rangle)^2|\psi\rangle = \langle(A - \langle A\rangle)^2\rangle. \tag{2}$$

In a more general case, if two Hermitian operators X, Y are canonically conjugate to each other (i.e., their commutator is the same as for the coordinate and momentum), then

$$\Delta X \Delta Y \geq \frac{\hbar}{2}. \tag{3}$$

Even more generally, for two arbitrary Hermitian operators A, B the following uncertainty relation is always valid:

$$\Delta A \Delta B \geq \frac{1}{2}|\langle\psi|[A, B]|\psi\rangle| \tag{4}$$

(the r.h.s. now depends on the system's state).

Some uncertainty relations are valid also in the situation of measurement. In this case the uncertainties included in the uncertainty relations have a quite different meaning. Let us consider this problem.

2.1. Formulation of the Measurement Uncertainty Relation

The following statement is often used in the quantum theory of measurements: if the coordinate q is measured with a finite precision, then the linear momentum p obtains, as a result of the measurement, an additional uncertainty determined by the Heisenberg's uncertainty relation. This statement deals with the two different states (before the measurement and after it) and may be illustrated by the following scheme:

$$\{\Delta q, \Delta p\} \stackrel{Dq}{\to} \{\delta q, \delta p\}, \quad \delta p \geq \frac{\hbar}{2Dq}. \tag{5}$$

The uncertainties Δq, Δp characterize the state before the measurement, δq, δp refer to the state after the measurement, and Dq characterizes the measurement precision (resolution). The statement claims that Dq and δp are connected by the Heisenberg's inequality. In other words, if q is measured with a resolution Dq, then the resulting uncertainty of p is not less than $\hbar/2Dq$.

For an arbitrary pair of canonically conjugate observables X and Y, an analogous statement should be valid:

$$\{\Delta X, \Delta Y\} \stackrel{DX}{\to} \{\delta X, \delta Y\}, \quad \delta Y \geq \frac{\hbar}{2DX}. \tag{6}$$

We shall see that relations of this type are indeed approximately valid. However, we shall consider also the situations where they are violated, and derive an inequality that is always valid.

2.2. Proof of the Approximate Measurement Uncertainty Relation

In order to prove the (approximate) Heisenberg's inequality (for definiteness, for q and p) in the situation of measurement, we have to accept some formal description of the processes occurring in the measurement. Let us describe the measurement, say, of the coordinate, by the scheme[1]

$$\psi(q) \stackrel{x}{\to} R(q-x)\psi(q) = \psi_x(q). \tag{7}$$

The initial state is described in this scheme by the wave function ψ in the q-representation. The final state ψ_x depends on the readout x of the measurement of the coordinate and is determined by the function $R(q)$ that characterizes the measurement, particularly, its precision (resolution). We shall assume that $R(q)$ is a "bell shape" positive function, though all the results will be valid also in the generic case.[2] The width of this function determines the precision of the measurement. The probability density of the measurement readout x is equal to the squared norm of the final state as defined in Eq. (7)

$$p(x) = ||\psi_x||^2. \tag{8}$$

This definition is illustrated in Fig. 1. The two vertical columns of diagrams in this figure present two different measurement readouts x. Depending on this readout, the initial wave function (represented in the upper diagrams by the function $P(q) = |\psi(q)|^2$) is converted after the measurement into one of the wave functions of the lower diagrams.

Let the uncertainties in the initial state ψ be Δq and Δp. The uncertainties in the final state ψ_x depend on x, but at the moment we shall denote them simply by δq and δp, not including x (in Sec. 2.3 we shall be forced to introduce more accurate notation).

To estimate the uncertainty δq, let us make use of the relations

$$\delta q^2 = \overline{(q-\overline{q})^2} = \min_a \overline{(q-a)^2} \leq \overline{(q-x)^2}$$

(with averaging over the state ψ_x) or, in a more explicit form,

$$\delta q^2 \leq \frac{\int (q-x)^2 R^2(q-x)|\psi(q)|^2}{\int R^2(q-x)|\psi(q)|^2} = \frac{\int q^2 R^2(q)|\psi(q+x)|^2}{\int R^2(q)|\psi(q+x)|^2}. \tag{9}$$

[1]This scheme is valid for a wide class of measurements, particularly, for the known (suggested already by von Neumann) model for the measurement of the coordinate q because of the interaction of the measured system with another system ("meter") due to the interaction Hamiltonian $H_I = g \cdot qP$ (where P is the linear momentum of the "meter").

[2]If $R(q)$ is not positive, then $R^2(q)$ must be replaced by $|R(q)|^2$ in all formulas, which does not change the arguments.

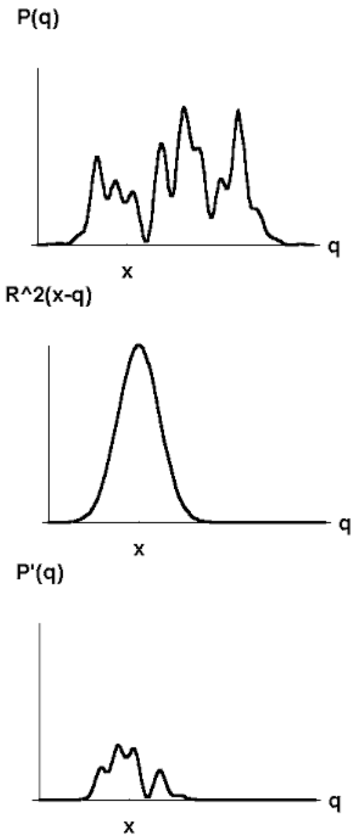

Fig. 1. Smooth projection of the wave function in a measurement of the coordinate. The initial wave function (upper row of diagrams) converts into the final function (lower row) obtained by multiplication by the characteristic function of the measurement. The resulting wave function depends on the readout of the measurement x (left- or right-hand vertical column of the diagrams).

We shall now take into account that the measurement's resolution is equal to Dq. This means that the "width" of the function $R(q)$ is Dq. This "width" may be defined in different ways. To make the proof simpler, we shall assume for the moment that the function $R(q)$ is symmetric and has a restricted support of width $2Dq$

$$\text{supp } R \subset [-Dq, Dq]. \tag{10}$$

This means that the function $R(q)$ is equal to zero outside the interval [– Dq, Dq]. Then the integrals in Eq. (9) may be taken in the limits $[-Dq, Dq]$

$$\delta q^2 \leq \frac{\int\limits_{|q|\leq Dq} q^2 R^2(q)|\psi(q+x)|^2}{\int\limits_{|q|\leq Dq} R^2(q)|\psi(q+x)|^2} \leq Dq^2.$$

This gives

$$\delta q \le Dq.$$

According to the usual Heisenberg uncertainty relation applied to the final (after the measurement) state of the measured system,

$$\delta q \, \delta p \ge \frac{\hbar}{2}.$$

Taking into account the preceding inequality, we have finally the measurement uncertainty relation

$$\delta p \ge \frac{\hbar}{2\delta q} \ge \frac{\hbar}{2Dq}. \qquad (11)$$

We have proved the required uncertainty assuming that the readout x of the measurement is precisely known. Strictly speaking, this is impossible because the spectrum of x is continuous. However, the result is also approximately valid if it is known that the measurement readout x lies in an interval of width $\Delta x \ll Dq$. A precise formulation of this statement is the following.

Let the measurement readout be known with some uncertainty, i.e., it may be represented by some probability density $p(x)$. Then the state of the system after the measurement must be described in a partly nonselective way [1]. In our case, this means that the final state is represented by the density matrix

$$\rho = \int dx \, p(x) \, \frac{|\psi_x\rangle\langle\psi_x|}{||\psi_x||^2}, \quad \int dx \, p(x) = 1.$$

If the readout is known with uncertainty Δx, then the width of the function $p(x)$ should be Δx. Assume that the support of the function $p(x)$ is restricted and has the width $2\Delta x$:

$$\text{supp } p \subset [a - \Delta x, a + \Delta x]. \qquad (12)$$

Then the final uncertainty of the linear momentum satisfies the inequality

$$\delta p_\rho \ge \frac{\hbar}{2(Dq + \Delta x)}.$$

The uncertainty is defined here as

$$\delta p_\rho^2 = \langle (p - \langle p \rangle_\rho)^2 \rangle_\rho$$

with the notation

$$\langle A \rangle_\rho = \text{tr}(\rho \, A).$$

The proof follows from the uncertainty relation for the mixed state

$$\delta p_\rho \delta q_\rho \ge \frac{\hbar^2}{4},$$

the inequality

$$\delta q_\rho^2 = \langle (q - \langle q \rangle_\rho)^2 \rangle_\rho \le \langle (q - a)^2 \rangle_\rho$$

(valid for arbitrary a), and the estimate

$$\int dx \, \frac{p(x)}{||\psi_x||^2} \int dq \, (q - a)^2 \, R^2(q - x) \, |\psi(q)|^2 \le (Dq + \Delta x)^2$$

following from Eqs. (10), (12).

There is no need to say that the above proof is straightforwardly generalized to the case of an arbitrary pair X, Y of canonically conjugate variables. In particular, if the momentum is measured with resolution Dp, we have the following relation for the uncertainty of the coordinate:

$$\{\Delta q, \Delta p\} \stackrel{Dp}{\to} \{\delta q, \delta p\}, \quad \delta q \geq \frac{\hbar}{2Dp}. \tag{13}$$

Starting from the more general Heisenberg's uncertainty relation (4), the measurement uncertainty for an arbitrary pair A, B of Hermitian operators may be proved. If the commutator of the operators is

$$[A, B] = i\, C,$$

then

$$\{\Delta A, \Delta B\} \stackrel{DA}{\to} \{\delta A, \delta B\}, \quad \delta B \geq \frac{|\langle C \rangle|}{2DA}, \tag{14}$$

where the mean value of C in the final state (after the measurement) must be substituted. This mean value depends on the measurement readout, so the general measurement uncertainty relation is not as convenient as for a c-number commutator.

In the preceding we have assumed that the characteristic function $R(q)$ of the measurement has a restricted support, i.e., is precisely null outside the interval $[-Dq, Dq]$. This assumption radically simplifies the proof, but it is not realistic. In reality, the function $R(q)$ may gradually die out but be nonzero far beyond the region where it has significant values. This is the case, for example, for the Gaussian characteristic function. The question naturally arises of whether the measurement uncertainty relation is applicable in the corresponding measurements or not.

An evident answer is the following.

A realistic function $R(q)$ may be replaced by another one coinciding with $R(q)$ in the domain where the latter has significant values but is precisely zero outside the domain. Then the formulated measurement uncertainty relation (11) may be considered as a good approximation for the real situation.

Yet it is possible to give a rigorous and physically precise solution of the problem as follows.

2.3. Precise Form of the Measurement-Induced Uncertainty Relation

Let the measurement's precision (resolution) be determined by the width of the characteristic function $R(q)$. Denote it by Dq and define it by

$$(Dq)^2 = \int q^2\, R^2(q)\, dq \tag{15}$$

instead of Eq. (10). This definition is constructed in analogy with the concept of the coordinate's uncertainty (notice that the function $R(q)$ is normalized according to $\int da\, |R|^2(a) = 1$). With this definition, the support of the function $R(q)$ may be much wider than of the width $2Dq$. It may even be infinitely wide. Now we shall consider measurement (7) with characteristic function $R(q)$ having these properties. Our task is to analyze the momentum uncertainty δp_x in the state $\psi_x(q)$ resulting in the measurement.

If the support of the function $R(q)$ is large, then the coordinate's uncertainty δq_x in the state $\psi_x(q)$ may also be large (compared with Dq) for some x. Therefore, despite the Heisenberg's uncertainty relation

$$\delta q_x\,\delta p_x \geq \frac{\hbar}{2}, \tag{16}$$

the uncertainty δp_x may be small (in comparison with $\hbar/2Dq$). However, this happens only for those values of x for which the probability density is low, so that these values of x practically never arise as measurement readouts.[3]

To express this in a more precise way, we shall average different values of $(1/\delta p_x)^2$ with the probability density corresponding to the values of x. Then the values of δp_x larger than $\hbar/2Dq$ will give a small contribution, so the resulting UR will be valid after averaging.

Recall that the probability density for x being a readout of the measurement is equal to the squared norm $||\psi_x||^2$ of the final state of the system [see Eq. (8)]. The measurement induced uncertainty relation may be written in the following form:

$$\int \frac{||\psi_x||^2}{\delta p_x^2}\,dx \leq \frac{4}{\hbar^2} Dq^2. \tag{17}$$

To prove this, write down the following chain of inequalities beginning with the Heisenberg's uncertainty relation (16)

$$\begin{aligned}\frac{\hbar^2}{4}\frac{1}{\delta p_x^2} \quad &\leq \quad \delta q_x^2 = \min_a \frac{\langle\psi_x|(q-a)^2|\psi_x\rangle}{||\psi_x||^2} \\ &\leq \quad \frac{\langle\psi_x|(q-x)^2|\psi_x\rangle}{||\psi_x||^2}.\end{aligned}$$

Now represent the mean value on the r.h.s. in an explicit form (as an integral over q). This gives

$$\frac{\hbar^2}{4}\frac{||\psi_x||^2}{\delta p_x^2} \leq \int (q-x)^2\, R^2(q-x)\, |\psi(q)|^2\, dq.$$

Integrating this inequality over x, assuming that the function $R(q)$ is symmetric and taking into account the definition (15), we have finally Eq. (17).

3. An Example of the Violation of the Heisenberg's Uncertainty Relation

We have seen that in the course of measurement of the coordinate the Heisenberg's inequality may be violated. This happens when the characteristic function of the measuring device has long tails. The violation occurs in the case of "nontypical" measurement readouts arising with small probability.

This is illustrated in Fig. 2. It shows how the wave function of the system (top diagram) changes in the case of typical (middle) and nontypical (bottom) measurement readout.

[3] As an example, let $\psi(q)$ consist of two equal peaks located far from each other symmetrically about $q = 0$, and the characteristic function $R(q)$ be Gaussian. Then the most probable values of x are those almost coinciding with the locations of the peaks. In these cases, the coordinate uncertainties δq_x will be small (of the order of the width of the peaks) and δp_x large, satisfying the measurement-induced uncertainty relation. However, x may be close to zero, although with exponentially small probability. Then δq_x will be equal to the distance between the peaks, i.e., very large.

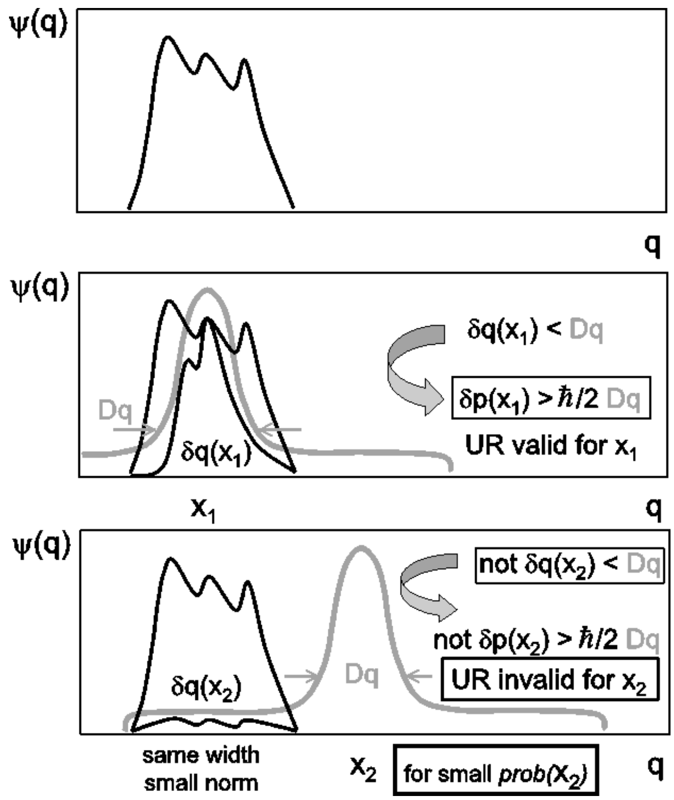

Fig. 2. Violation of the Heisenberg's inequality in measurement characterized by a function with long tails.

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