

# The statistical origins of gauge coupling and spin

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## Abstract

A previous one-dimensional derivation of Schrödinger's equation from statistical assumptions is generalized to three spatial dimensions, gauge fields, and spin. It is found that the same statistical assumptions that imply Schrödinger's equation determine also the form of the gauge coupling terms, and the form of the corresponding local (Lorentz) forces. An explanation for the role of the electrodynamic potentials, as statistical representatives of the Lorentz force, is given. Spin one-half is introduced as the property of a statistical ensemble to respond to an external gauge field in two different ways. A generalized calculation, using the twofold number of variables, leads to Pauli's equation. The new spin term is again the statistical representative of the corresponding local force. The classical limit  $\hbar \rightarrow 0$  of Schrödinger's equation and closely related questions of interpretation of the quantum mechanical formalism are discussed.

## 1 Introduction

This paper is a sequel to a previous work [18] of the present author, which will be referred to as I. In I an attempt has been undertaken to construct a new 'statistical' approach to quantum theory. This approach is based on the idea that quantum mechanics is not a theory about particles but about statistical ensembles. It is well known that the dynamic numerical output of quantum mechanics consists of probabilities. A probability is a "deterministic" prediction which can be verified in a statistical sense only, i.e. by performing experiments on a large number of identically prepared individual systems [7],[24]. Therefore, quantum mechanics is a theory about statistical ensembles [5] and can only be used to make predictions about individual events if additional intellectual constructs, which are not part of the physical formalism, are introduced. The work reported in I, as well as the present one, can be characterized by complete absence of such constructs.

Adopting this point of view one is immediately led to the idea that a quantization procedure should exist, which is based on statistical concepts and assumptions. It must, of course, lead to the same results as the standard (canonical) quantization method but should be based on physically interpretable assumptions. In this respect, the 'classical' canonical quantization procedure, which is based on the single-particle picture, is not satisfying because it consists of a number of purely formal rules. From a positivistic point of view all quantization methods leading to the same final result are equivalent. From the present point of view this is not the case and comprehensibility matters.

The quantization method reported in I is essentially based on the validity of the following three assumptions: (i) two differential equations which are similar in structure to the canonical equations of classical mechanics but with observables replaced by expectation values, (ii) a local conservation law of probability with a particular form of the probability current, and (iii) a differential version (minimal Fisher information) of the statistical principle of maximal disorder. As has been shown in I these postulates imply Schrödinger's equation "for a single particle" (i.e. for an ensemble of identically prepared single particles) in an external mechanical potential. This derivation provides a statistical explanation for the ad hoc rules of the conventional (single-particle) canonical quantization method. The treatment in I was restricted to a single spatial dimension.

In the present paper the work reported in I is extended to three spatial dimensions, gauge fields and spin. In section 2 the fundamental ideas are reviewed and the basic equations of the three-dimensional

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theory are listed. In section 3 an integral relation which provides a basis for later calculation is derived. In the central section 4 we pose the following question: Which constraints on admissible forces exist for the present class of statistical theories ? The answer is that only macroscopic forces of the form of the Lorentz force can occur in nature. These forces are statistically represented by potentials, i.e. by the familiar gauge coupling terms in matter field equations. The present statistical approach provides a natural explanation for the long-standing question why potentials play an indispensable role in the field equations of physics. In section 5 it is shown that among all statistical theories only the time-dependent Schrödinger equation follows the logical requirement of maximal disorder or minimal Fisher information. In section 6 the basic equations for a generalized theory, using the double number of dynamic variables, are formulated. The final result obtained in section 7 is Pauli's equation for a spin ensemble of particles. In section 8 the classical limit of quantum mechanics is studied and the misleading role of the principle of reductionism is pointed out. In Section 9 related questions concerning the role of potentials and the general interpretation of quantum mechanics are discussed. A comparison of electro-dynamical and inertial gauge fields is reported in appendix A. The final Section 10 contains concluding remarks.

## 2 Basic equations

In I three different types of theories have been defined which differ from each other with regard to the role of probability. The dogma underlying theories of type 1 is determinism with regard to single events; probability does not play any role. If nature behaves according to this dogma, then measurements on identically prepared individual systems yield identical results. Classical mechanics is obviously such a deterministic type 1 theory. We shall use below (as a 'template' for the dynamics of our statistical theories) the following version of Newton's law, where the particle momentum  $p_k(t)$  plays the role of a second dynamic variable besides the spatial coordinate  $x_k(t)$ :

$$\frac{d}{dt}x_k(t) = \frac{p_k(t)}{m}, \quad \frac{d}{dt}p_k(t) = F_k(x, p, t). \quad (1)$$

In classical mechanics there is no restriction as regards the admissible forces. Thus,  $F_k$  is an arbitrary function of  $x_k, p_k, t$ ; it is, in particular, not required that it be derivable from a potential.

Experimental data from atomic systems indicate that nature does not behave according to this single-event deterministic dogma. A simple but somewhat unfamiliar idea is, to construct a theory which is deterministic only in a statistical sense. This means that measurements on identically prepared individual systems do not yield identical results (no determinism with regard to single events) but repeated measurements on ensembles [consisting each time of a large (infinite) number of measurements on individual systems] yield identical results. In this case we have determinism with regard to ensembles (expectation values, or probabilities).

Note that such a theory is still far from chaotic even if our macroscopic anticipation of (single-event) determinism is not satisfied. Note also that there is no reason to assume that such a statistical theory for microscopic events is incompatible with macroscopic determinism. It is a frequently observed (but not completely understood) phenomenon in nature that systems with many (microscopic) degrees of freedom can be described by a much smaller number of variables. During this process of elimination of variables the details of the corresponding microscopic theory for the individual constituents are generally lost. In other words, there is no reason to assume that a fundamental statistical law for individual atoms and a deterministic law for a piece of matter consisting of, say,  $10^{23}$  atoms should not be compatible with each other. This way of characterizing a relation between two different physical theories differs from the conventional reductionistic point of view but similar positions may be found in the literature [3],[23].

As discussed in I two types (referred to as type 2 and type 3) of indeterministic theories may be identified. In type 2 theories laws for individual particles exist (roughly speaking the individuality of particles remains intact) but the initial values are unknown and are described by probabilities only. An example for such a (classical-statistical) type 2 theory is statistical thermodynamics. On the other hand, in type 3 theories the amount of uncertainty is still greater, insofar as no dynamic laws for individual particles exist any more. A possible candidate for this 'extreme' type of indeterministic theory is quantum mechanics. The method used in I to construct statistical theories is based on three assumptions listed in the last section. The first and second of these cover type 2 as well as type 3 theories, while the third - the requirement of maximal disorder - does only hold for a single type 3 theory, namely quantum mechanics. In this sense quantum mechanics may be considered as the most reasonable theory among all statistical theories defined by the first two assumptions. There is obviously an analogy between quantum mechanics and the principle of minimal Fisher information on the one hand and classical statistical mechanics and

the principle of maximal entropy on the other hand; both theories are realizations of the principle of maximal disorder.

The basic equations of I (see section 3 of I) are generalized with respect to the number of spatial dimensions and with respect to the structure of the function  $S(x, t)$ . In I  $S(x, t)$  was an 'ordinary' - i.e. single-valued - function. Now, we allow for multi-valued functions  $\tilde{S}(x, t)$ . This is possible because  $\tilde{S}(x, t)$  itself does not appear in any physical law (see below) of the present theory. It will be shown that this 'degree of freedom' is intimately related to the existence of gauge fields. A multi-valued  $\tilde{S}(x, t)$  cannot be an observable quantity. However, all quantities derived from  $\tilde{S}(x, t)$ , which occur in physical laws must be observables and must be single valued. Of particular importance are the first derivatives with respect to  $t$  and  $x_k$ . We assume that  $\tilde{S}(x, t)$  may be written as a sum of a single-valued part  $S(x, t)$  and a multi-valued part  $\tilde{N}$ . Then, given that

$$\tilde{S}(x, t) \text{ multi-valued, } \quad \frac{\partial \tilde{S}}{\partial t}, \frac{\partial \tilde{S}}{\partial x_k} \text{ single-valued,} \quad (2)$$

the derivatives of  $\tilde{S}(x, t)$  may be written in the form

$$\frac{\partial \tilde{S}}{\partial t} = \frac{\partial S}{\partial t} + e\Phi, \quad \frac{\partial \tilde{S}}{\partial x_k} = \frac{\partial S}{\partial x_k} - \frac{e}{c}A_k, \quad (3)$$

where the four functions  $\Phi$  and  $A_k$  are proportional to the derivatives of the multi-valued part  $\tilde{N}$  with respect to  $t$  and  $x_k$  respectively (Note the change in sign of  $\Phi$  and  $A_k$  in comparison to [19]; this is due to the fact that the multi-valued phase is now denoted by  $\tilde{S}$ ). The physical motivations for introducing the pre-factors  $e$  and  $c$  in Eq. (3) have been extensively discussed elsewhere [17],[19] in an electro-dynamical context.

The necessary and sufficient condition for single-valuedness of a function  $\tilde{H}(x, t)$  (in a subspace  $\mathcal{G} \subseteq \mathcal{R}^4$ ) is that all second order derivatives of  $\tilde{H}(x, t)$  with respect to  $x_k$  and  $t$  commute with each other (in  $\mathcal{G}$ ). In this sense  $\tilde{S}(x, t)$  is multi-valued while the four derivatives of  $S(x, t)$  with respect to  $x_k$  and  $t$  and the four functions  $\Phi$  and  $A_k$  are single-valued. On the other hand this does not mean that the latter eight quantities must be unique. Actually it will turn out that they are not; according to the present construction only the four derivatives of  $\tilde{S}(x, t)$  with respect to  $x_k$  and  $t$  are uniquely determined by the physical situation. These derivatives define four fields

$$\tilde{p}_k(x, t) = \frac{\partial \tilde{S}(x, t)}{\partial x_k}, \quad \tilde{E}(x, t) = -\frac{\partial \tilde{S}(x, t)}{\partial t}, \quad (4)$$

with dimensions of momentum and energy respectively (a quantity denoted  $\tilde{A}$  is not necessarily multi-valued; this notation is used here to indicate that it is defined with the help of a multi-valued  $\tilde{S}$ ).

In contrast to  $\tilde{S}$ , our second fundamental dynamic variable  $\rho$  is a physical observable (in the statistical sense) and is treated as a single-valued function. The fields  $S$  and  $\rho$  (we use the summation convention) obey the continuity equation

$$\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial}{\partial x_k} \frac{\rho(x, t)}{m} \frac{\partial \tilde{S}(x, t)}{\partial x_k} = 0 \quad (5)$$

The statistical conditions associated with the type 1 theory (1), are obtained in the same way as in I by replacing the observables  $x_k(t)$ ,  $p_k(t)$  and the force field  $F_k(x(t), p(t), t)$  by averages  $\overline{x_k}$ ,  $\overline{p_k}$  and  $\overline{F_k}$ . This leads to the relations

$$\frac{d}{dt} \overline{x_k} = \frac{\overline{p_k}}{m} \quad (6)$$

$$\frac{d}{dt} \overline{p_k} = \overline{F_k(x, p, t)}, \quad (7)$$

where the averages are given by the following integrals over the random variables  $x_k$ ,  $p_k$  [which should be distinguished from the observables  $x_k(t)$ ,  $p_k(t)$ ]:

$$\overline{x_k} = \int_{-\infty}^{\infty} d^3x \rho(x, t) x_k \quad (8)$$

$$\overline{p_k} = \int_{-\infty}^{\infty} d^3p w(p, t) p_k \quad (9)$$

$$\overline{F_k(x, p, t)} = \int_{-\infty}^{\infty} d^3x d^3p W(x, p, t) F_k(x, p, t). \quad (10)$$

The time-dependent probability densities  $W$ ,  $\rho$ ,  $w$  are positive semidefinite and normalized to unity, i.e. they fulfill the conditions

$$\int_{-\infty}^{\infty} d^3x \rho(x, t) = \int_{-\infty}^{\infty} d^3p w(p, t) = \int_{-\infty}^{\infty} d^3x d^3p W(x, p, t) = 1 \quad (11)$$

The densities  $\rho$  and  $w$  may be derived from the fundamental probability density  $W$  by means of the relations

$$\rho(x, t) = \int_{-\infty}^{\infty} d^3p W(x, p, t); \quad w(p, t) = \int_{-\infty}^{\infty} d^3x W(x, p, t). \quad (12)$$

The present construction of the statistical conditions (6) and (7) from the type 1 theory (1) is very similar to the treatment in I. There are, however, two differences. The first is that we allow now for a  $p$ -dependent external force. This leads to a more complicated probability density  $W(x, p, t)$  as compared to the two decoupled densities  $\rho(x, t)$  and  $w(p, t)$  of I. The second difference, which is in fact related to the first, is the use of a multi-valued  $\tilde{S}(x, t)$ .

### 3 Statistical conditions

In this section we combine and rewrite the statistical conditions in order to obtain a single relation that will be useful in later sections .

We begin with the first statistical condition (6). The following calculation is very similar to the corresponding one-dimensional calculation reported in I; thus details may be omitted. We insert the definition (8) in (6) and replace the derivative of  $\rho$  with respect to  $t$  by the second term in the continuity equation (5). Using Gauss' integral theorem and assuming that  $\rho$  vanishes sufficiently rapidly for  $|x| \rightarrow \infty$  in order for the surface integral to vanish (we may even assume that it vanishes faster than an arbitrary finite power of  $x_k^{-1}$ ) we arrive at the following expression for the expectation value of the momentum

$$\int_{-\infty}^{\infty} d^3x \rho(x, t) \frac{\partial \tilde{S}(x, t)}{\partial x_k} = \bar{p}_k. \quad (13)$$

This relation agrees essentially with the corresponding one-dimensional relation obtained in I.

We study now the implications of the second statistical condition (7). We start by evaluating the left hand side of (7). Using the variables  $\rho$ ,  $\tilde{S}$  it is given by

$$\frac{d}{dt} \bar{p}_k = \frac{d}{dt} \int_{-\infty}^{\infty} d^3x \rho(x, t) \frac{\partial \tilde{S}(x, t)}{\partial x_k}. \quad (14)$$

Performing the derivative with respect to  $t$  Eq. (14) takes the form

$$\frac{d}{dt} \bar{p}_k = \int_{-\infty}^{\infty} d^3x \left[ \frac{\partial \rho(x, t)}{\partial t} \frac{\partial \tilde{S}(x, t)}{\partial x_k} + \rho(x, t) \frac{\partial}{\partial t} \frac{\partial \tilde{S}(x, t)}{\partial x_k} \right]. \quad (15)$$

Note that each term in the integrand of (15) is single-valued but  $\tilde{S}$  is not. As a consequence the order of two derivatives of  $\tilde{S}$  (with respect to anyone of the variables  $x_k, t$ ) must not be changed. We introduce the (single-valued) quantities

$$\tilde{S}_{[j,k]} = \left[ \frac{\partial^2 \tilde{S}}{\partial x_j \partial x_k} - \frac{\partial^2 \tilde{S}}{\partial x_k \partial x_j} \right], \quad \tilde{S}_{[0,k]} = \left[ \frac{\partial^2 \tilde{S}}{\partial t \partial x_k} - \frac{\partial^2 \tilde{S}}{\partial x_k \partial t} \right] \quad (16)$$

to describe the non-commuting derivatives.

Evaluating the first term in the integrand of (15) we replace the time derivative of  $\rho$  by the divergence of the probability current according to the continuity equation (5) to obtain

$$\int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial t} \frac{\partial \tilde{S}}{\partial x_k} = - \int_{-\infty}^{\infty} d^3x \frac{\partial \tilde{S}}{\partial x_k} \frac{\partial}{\partial x_j} \frac{\rho}{m} \frac{\partial \tilde{S}}{\partial x_j}. \quad (17)$$

Performing a partial integration and exchanging the derivatives with respect to  $x_k$  and  $x_j$ , Eq. (17) takes the form

$$\int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial t} \frac{\partial \tilde{S}}{\partial x_k} = \int_{-\infty}^{\infty} d^3x \frac{\rho}{m} \frac{\partial \tilde{S}}{\partial x_j} \left[ \frac{\partial}{\partial x_k} \frac{\partial \tilde{S}}{\partial x_j} + \tilde{S}_{[j,k]} \right]. \quad (18)$$

Using the formula

$$2 \frac{\partial \tilde{S}}{\partial x_j} \frac{\partial^2 \tilde{S}}{\partial x_k \partial x_j} = \frac{\partial}{\partial x_k} \sum_j \left( \frac{\partial \tilde{S}}{\partial x_j} \right)^2, \quad (19)$$

and performing a second partial integration the first term in the integrand of (15) takes the form

$$\begin{aligned} & \int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial t} \frac{\partial \tilde{S}}{\partial x_k} = \\ & - \int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial x_k} \frac{1}{2m} \sum_j \left( \frac{\partial \tilde{S}}{\partial x_j} \right)^2 + \int_{-\infty}^{\infty} d^3x \frac{\rho}{2m} \frac{\partial \tilde{S}}{\partial x_j} \tilde{S}_{[j,k]}, \end{aligned} \quad (20)$$

Similar manipulations lead to the following expression for the second term in the integrand of (15):

$$\int_{-\infty}^{\infty} d^3x \rho \frac{\partial}{\partial t} \frac{\partial \tilde{S}}{\partial x_k} = - \int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial x_k} \frac{\partial \tilde{S}}{\partial t} + \int_{-\infty}^{\infty} d^3x \rho \tilde{S}_{[0,k]}, \quad (21)$$

Let us assume that the macroscopic force  $F_k(x, p, t)$  entering the second statistical condition (7) can be written as a sum of two contributions,  $F_k^{(m)}(x, t)$  and  $F_k^{(e)}(x, p, t)$ ,

$$F_k(x, p, t) = F_k^{(m)}(x, t) + F_k^{(e)}(x, p, t), \quad (22)$$

where  $F_k^{(m)}(x, t)$  takes the form of a negative gradient of a scalar function  $V(x, t)$  (mechanical potential). Since  $F_k^{(m)}$  does (in contrast to  $F_k^{(e)}$ ) not depend on  $p$ , its average value can be calculated with the help of a known probability distribution, namely the dynamical variable  $\rho$ . Performing a partial integration and collecting terms the second statistical condition (7) takes the form.

$$\begin{aligned} & - \int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial x_k} \left[ \frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \sum_j \left( \frac{\partial \tilde{S}}{\partial x_j} \right)^2 + V \right] \\ & + \int_{-\infty}^{\infty} d^3x \rho \left[ \frac{1}{m} \frac{\partial \tilde{S}}{\partial x_j} \tilde{S}_{[j,k]} + \tilde{S}_{[0,k]} \right] = \overline{F_k^{(e)}(x, p, t)} \end{aligned} \quad (23)$$

Comparing Eq. (23) with the corresponding formula obtained in I [see Eq. (24) of I] we see that two new terms appear now, namely the expectation value of the  $p$ -dependent force on the r.h.s. and the second term on the l.h.s. of Eq. (23). The latter is a direct consequence of our assumption of a multi-valued variable  $\tilde{S}$ .

In the next section it will be shown that for vanishing multi-valuedness Eq. (23) has to agree with the three-dimensional generalization of the corresponding result [Eq. (24) of I] obtained in I. This means that the  $p$ -dependent term on the r.h.s. has to vanish too in this limit and indicates a relation between multi-valuedness of  $\tilde{S}$  and  $p$ -dependence of the external force.

## 4 Gauge coupling

At this stage of our study it may be useful to clarify the physical role of the quantities  $\tilde{S}$ ,  $S$ , and  $A_k, \Phi$  introduced at the beginning of section 2. The multi-valued function  $\tilde{S}$  may be represented [17],[19] in the form

$$\tilde{S}(x, t; \mathcal{C}) = S(x, t) - \frac{e}{c} \int_{x_0, t_0; \mathcal{C}}^{x, t} [dx'_k A_k(x', t') - c dt' \phi(x', t')], \quad (24)$$

as a path-integral performed along an arbitrary path  $\mathcal{C}$  in four-dimensional space; the multi-valuedness of  $\tilde{S}$  means simply that it depends not only on  $x, t$  but also on the path  $\mathcal{C}$  connecting the points  $x_0, t_0$  and  $x, t$ .

The quantity  $\tilde{S}$  cannot be a physical observable because of its multi-valuedness. The fundamental physical quantities to be determined by our (future) theory are the four derivatives of  $\tilde{S}$  [see (3)] which will be rewritten here using the above notation

$$\frac{\partial \tilde{S}(x, t; \mathcal{C})}{\partial t} = \frac{\partial S(x, t)}{\partial t} + e\Phi(x, t), \quad (25)$$

$$\frac{\partial \tilde{S}(x, t; \mathcal{C})}{\partial x_k} = \frac{\partial S(x, t)}{\partial x_k} - \frac{e}{c} A_k(x, t), \quad (26)$$

But we encounter here a somewhat unusual situation: On the one hand the left hand sides of (25) (26) are the basic physical quantities of our theory, on the other hand we cannot solve our (future) differential equations for these quantities because of the peculiar multi-valued structure of  $\vec{S}$ . We have to use instead the decompositions as given by the right hand sides of (25) and (26). The latter terms are single-valued (in the sense of the above definition) but need not be unique because only the left hand sides are uniquely determined by the physical situation. We tentatively assume that the fields  $\Phi$  and  $A_k$  are 'given' quantities in the sense that they represent an external influence (of 'external forces') on the considered statistical situation. An actual calculation has to be performed in such a way that fixed fields  $\Phi$  and  $A_k$  are chosen and then the differential equations are solved for  $S$  (and  $\rho$ ). However, as mentioned already, what is actually uniquely determined by the physical situation is the *sum* of the two terms on the right hand sides of (25) (26). Consequently, a different set of fixed fields  $\Phi'$  and  $A'_k$  may lead to a different solution  $S'$  in such a way that the sum of the new terms [on the right hand sides of (25) and (26)] is the *same* as the sum of the old terms. We assume here, that the formalism restores the values of the physically relevant terms. This implies that the relation between the old and new terms is given by

$$S'(x, t) = S(x, t) + \varphi(x, t) \quad (27)$$

$$\Phi'(x, t) = \Phi(x, t) - \frac{1}{e} \frac{\partial \varphi(x, t)}{\partial t} \quad (28)$$

$$A'_k(x, t) = A_k(x, t) + \frac{c}{e} \frac{\partial \varphi(x, t)}{\partial x_k}, \quad (29)$$

where  $\varphi(x, t)$  is an arbitrary, single-valued function of  $x_k, t$ . Consequently, all 'theories' (differential equations for  $S$  and  $\rho$  defined by the assumptions listed in section 2) will be form-invariant under the transformations (27)-(29). These invariance transformations are (using an arbitrary function  $\chi = c\varphi/e$  instead of  $\varphi$ ) denoted as 'gauge transformations of the second kind'.

The fields  $\Phi(x, t)$  and  $A_k(x, t)$  describe an external influence but their numerical value is undefined; their value at  $x, t$  may be changed according to (28) and (29) without changing their physical effect. Thus, these fields cannot play a local role in space and time like forces and fields in classical mechanics and electrodynamics. What, then, is the physical meaning of these fields? An explanation which seems obvious in the present context is the following: They describe the *statistical effect* of an external influence on the considered system (ensemble of identically prepared individual particles). The statistical effect of a force field on an ensemble may obviously differ from the local effect of the same force field on individual particles; thus the very existence of fields  $\Phi$  and  $A_k$  different from  $\vec{E}$  and  $\vec{B}$  is no surprise. The second common problem with the interpretation of the potentials  $\Phi$  and  $A_k$  is their non-uniqueness. It is hard to understand that a quantity ruling the behavior of individual particles should not be uniquely defined. In contrast, this non-uniqueness is much easier to accept if  $\Phi$  and  $A_k$  rule the behavior of ensembles instead of individual particles. We have no problem to accept the fact that a function that represents a global (integral) effect may have many different local realizations.

It seems that this interpretation of the 'potentials'  $\Phi$  and  $A_k$  is highly relevant for the interpretation of the Aharonov-Bohm effect [1], [27]. A statistical interpretation of the potentials has apparently never been suggested, neither in the vast literature on the Aharonov-Bohm effect nor in papers promoting the statistical interpretation of quantum mechanics; most physicists discuss this nonlocal 'paradox' from the point of view of "the wave function of a single electron". Further discussion on the significance of the potentials may be found in [section 9](#).

The expectation value  $F_k^{(e)}(x, p, t)$  on the right hand side of (23) is to be calculated using local, macroscopic forces (whose functional form is still unknown). Both the potentials and these local forces represent an external influence, and it is plausible to assume that they are not independent from each other. Thus it is reasonable to assume that the (nonlocal) potentials are statistical representations of the *same* external (local) forces, occurring on the r.h.s. of Eq. (23). These local forces have to be determined by the potentials but must be uniquely defined at each space-time point. The gauge-invariant fields

$$E_k = -\frac{1}{c} \frac{\partial A_k}{\partial t} - \frac{\partial \Phi}{\partial x_k}, \quad B_k = \epsilon_{kij} \frac{\partial A_j}{\partial x_i}, \quad (30)$$

fulfill these requirements. As a consequence of the defining relations (30) they obey automatically the homogeneous Maxwell equations.

$$\frac{\partial \vec{B}}{\partial \vec{r}} = 0, \quad \frac{\partial}{\partial \vec{r}} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0. \quad (31)$$

Note that from the present statistical (nonlocal) point of view the potentials are more fundamental than the local fields. In contrast, considered from the point of view of macroscopic physics, the local fields are the physical quantities of primary importance and the potentials may (or may not) be introduced for mathematical convenience.

As a next step we rewrite the second term on the l.h.s. of Eq. (23). The commutator terms (16) take the form

$$\tilde{S}_{[0,k]} = -e \left( \frac{1}{c} \frac{\partial A_k}{\partial t} + \frac{\partial \Phi}{\partial x_k} \right), \quad \tilde{S}_{[j,k]} = \frac{e}{c} \left( \frac{\partial A_j}{\partial x_k} - \frac{\partial A_k}{\partial x_j} \right). \quad (32)$$

As a consequence, they may be expressed in terms of the local fields (30), which have been introduced above for reasons of gauge-invariance. Using (32),(30) and the relation (4) for the momentum field, Eq. (23) takes the form

$$\begin{aligned} & - \int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial x_k} \left[ \frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \sum_j \left( \frac{\partial \tilde{S}}{\partial x_j} \right)^2 + V \right], \\ & + \int_{-\infty}^{\infty} d^3x \rho \left[ \frac{e}{c} \epsilon_{kij} \tilde{v}_i B_j + e E_k \right] = \overline{F_k^{(e)}(x, p, t)} \end{aligned} \quad (33)$$

with a velocity field  $\tilde{v}_i$  defined by

$$\tilde{v}_i(x, t) = \frac{1}{m} \frac{\partial \tilde{S}(x, t)}{\partial x_i}. \quad (34)$$

Thus, the new terms on the l.h.s. of (33) - stemming from the multi-valuedness of  $\tilde{S}$  - take the form of an expectation value (with  $\mathcal{R}^3$  as sample space) of the Lorentz force field

$$\vec{F}_L(x, t) = e \vec{E}(x, t) + \frac{e}{c} \vec{v}(x, t) \times \vec{B}(x, t), \quad (35)$$

if the particle velocity is identified with the velocity field (34).

Let us discuss now the nature of the macroscopic forces  $F_k^{(e)}(x, p, t)$  entering the expectation value on the r.h.s. of Eq. (33). In our type I parent theory, classical mechanics, there are no constraints for the possible functional form of  $F_k^{(e)}(x, p, t)$ . However, this need not be true in the present statistical framework. As a matter of fact, the way the mechanical potential  $V(x, t)$  entered the differential equation for  $S$  (in the previous work I) indicates already that such constraints do actually exist. Let us recall that we tacitly restricted the class of forces to those derivable from a potential  $V(x, t)$ . If we eliminate this restriction and admit arbitrary forces, with components  $F_k(x, t)$ , we obtain instead of the above relation (33) the simpler relation [Eq. (24) of I, generalized to three dimensions and arbitrary forces of the form  $F_k(x, t)$ ]

$$- \int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial x_k} \left[ \frac{1}{2m} \sum_j \left( \frac{\partial S}{\partial x_j} \right)^2 + \frac{\partial S}{\partial t} \right] = \int_{-\infty}^{\infty} dx \rho F_k(x, t). \quad (36)$$

This is a rather complicated integro-differential equation for our variables  $\rho(x, t)$  and  $S(x, t)$ . We assume now, using mathematical simplicity as a guideline, that Eq. (36) can be written in the common form of a local differential equation. This assumption is of course not evident; in principle the laws of physics could be integro-differential equations or delay differential equations or take an even more complicated mathematical form. Nevertheless, this assumption seems rather weak considering the fact that all fundamental laws of physics take this 'simple' form. Thus, we postulate that Eq. (36) is equivalent to a differential equation

$$\frac{1}{2m} \sum_j \left( \frac{\partial S}{\partial x_j} \right)^2 + \frac{\partial S}{\partial t} + T = 0, \quad (37)$$

where the unknown term  $T$  describes the influence of the force  $F_k$  but may also contain other contributions. Let us write

$$T = -L_0 + V, \quad (38)$$

where  $L_0$  does not depend on  $F_k$ , while  $V$  depends on it and vanishes for  $F_k \rightarrow 0$ . Inserting (37) and (38) in (36) yields

$$\int d^3x \frac{\partial \rho}{\partial x_k} (-L_0 + V) = \int d^3x \rho F_k(x, t). \quad (39)$$

For  $F_k \rightarrow 0$  Eq. (39) leads to the relation

$$\int d^3x \frac{\partial \rho}{\partial x_k} L_0 = 0, \quad (40)$$

which remains true for finite forces because  $L_0$  does not depend on  $F_k$ . Finally, performing a partial integration, we see that a relation

$$F_k = -\frac{\partial V}{\partial x_k} + s_k, \quad \int_{-\infty}^{\infty} d^3x \rho s_k = 0, \quad (41)$$

exists between  $F_k$  and  $V$ , with a vanishing expectation value of the (statistically irrelevant) functions  $s_k$ . This example shows that the restriction to gradient fields, made above and in I, is actually not necessary. We may *admit* force fields which are arbitrary functions of  $x$  and  $t$ ; the statistical conditions (which play now the role of a 'statistical constraint') eliminate automatically all forces that cannot be written after statistical averaging as gradient fields.

This is very interesting and indicates the possibility that the present statistical assumptions leading to Schrödinger's equation may also be responsible, at least partly, for the structure of the real existing (gauge) interactions of nature.

This statistical constraint may also work in the present  $p$ -dependent case. We assume that the force in (33) is a standard random variable with the configuration space as sample space (see the discussion in section 4 of I) and that the variable  $p$  in  $F_k^{(e)}(x, p, t)$  may consequently be replaced by the field  $\tilde{p}(x, t)$  [see (4)]. Then, the expectation value on the r.h.s. of (33) takes the form

$$\overline{F_k^{(e)}(x, p, t)} = \int_{-\infty}^{\infty} d^3x \rho(x, t) H_k(x, \frac{\partial \tilde{S}(x, t)}{\partial x}, t). \quad (42)$$

The second term on the l.h.s. of (33) has the *same* form. Therefore, the latter may be eliminated by writing

$$H_k(x, \frac{\partial \tilde{S}}{\partial x}, t) = \frac{e}{c} \epsilon_{kij} \frac{1}{m} \frac{\partial \tilde{S}}{\partial x_i} B_j + e E_k + h_k(x, \frac{\partial \tilde{S}}{\partial x}, t), \quad (43)$$

with  $h_k(x, p, t)$  as our new unknown functions. They obey the simpler relations

$$-\int_{-\infty}^{\infty} d^3x \frac{\partial \rho}{\partial x_k} \left[ \frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \sum_j \left( \frac{\partial \tilde{S}}{\partial x_j} \right)^2 + V \right] = \int_{-\infty}^{\infty} d^3x \rho h_k(x, \frac{\partial \tilde{S}}{\partial x}, t). \quad (44)$$

On a first look this condition for the allowed forces looks similar to the  $p$ -independent case [see (36)]. But the dependence of  $h_k$  on  $x, t$  cannot be considered as 'given' (externally controlled), as in the  $p$ -independent case, because it contains now the unknown  $x, t$ -dependence of the derivatives of  $\tilde{S}$ . We may nevertheless try to incorporate the r.h.s. by adding a term  $\tilde{T}$  to the bracket which depends on the derivatives of the multivalued quantity  $\tilde{S}$ . This leads to the condition

$$h_k(x, \frac{\partial \tilde{S}}{\partial x}, t) = -\frac{\partial \tilde{T}(x, \frac{\partial \tilde{S}}{\partial x}, t)}{\partial x_k} + s_k, \quad \int_{-\infty}^{\infty} d^3x \rho s_k = 0. \quad (45)$$

But this relation cannot be fulfilled for nontrivial  $h_k, \tilde{T}$  because the derivatives of  $\tilde{S}$  cannot be subject to further constraints beyond those given by the differential equation; on top of that the derivatives with regard to  $x$  on the r.h.s. create higher order derivatives of  $\tilde{S}$  which are not present at the l.h.s. of Eq. (45). The only possibility to fulfill this relation is for constant  $\frac{\partial \tilde{S}}{\partial x}$ , a special case which has in fact already been taken into account by adding the mechanical potential  $V$ . We conclude that the statistical constraint leads to  $h_k = \tilde{T} = 0$  and that the statistical condition (44) takes the form

$$-\int d^3x \frac{\partial \rho}{\partial x_k} \left[ \frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \sum_j \left( \frac{\partial \tilde{S}}{\partial x_j} \right)^2 + V \right] = 0. \quad (46)$$

Thus, only a mechanical potential and the four electrodynamic potentials are compatible with the statistical constraint and will consequently - assuming that the present statistical approach reflects a fundamental principle of nature - be realized in nature. As is well known all existing interactions follow



(sometimes in a generalized form) the gauge coupling scheme derived above. The statistical conditions imply not only Schrödinger's equation but also the form of the (gauge) coupling to external influences and the form of the corresponding local force, the Lorentz force.

In the present derivation the usual order of proceeding is just inverted. In the conventional deterministic treatment the form of the local forces (Lorentz force), as taken from experiment, is used as a starting point. The potentials are introduced afterwards, in the course of a transition to a different formal framework (Lagrange formalism). In the present approach the fundamental assumptions are the statistical conditions. Then, taking into account an existing mathematical freedom (multi-valuedness of a variable) leads to the introduction of potentials. From these, the shape of the macroscopic (Lorentz) force can be derived, using the validity of the statistical conditions as a constraint.

## 5 Fisher information

In this section a local differential equation for  $S$  and  $\rho$  will be derived from the integral equation (46). As our essential constraint we will use, besides general principles of simplicity (like homogeneity and isotropy of space) the principle of maximal disorder, as realized by the requirement of minimal Fisher information. Using the abbreviation

$$\tilde{L}(x, t) = \frac{\partial \tilde{S}}{\partial t} + \frac{1}{2m} \left( \frac{\partial \tilde{S}(x, t)}{\partial x} \right)^2 + V(x, t), \quad (47)$$

the general solution of (46) may be written in the form

$$\frac{\partial \rho}{\partial x_k} \tilde{L}(x, t) = G_k(x, t), \quad (48)$$

where the three functions  $G_k(x, t)$  have to vanish upon integration over  $\mathcal{R}^3$  and are otherwise arbitrary. If we restrict ourselves to an isotropic law, we may write

$$G_k(x, t) = \frac{\partial \rho}{\partial x_k} L_0. \quad (49)$$

Then, our problem is to find a function  $L_0$  which fulfills the differential equation

$$\tilde{L}(x, t) - L_0 = 0, \quad (50)$$

and condition (40). The method used in I for a one-dimensional situation, to determine  $L_0$  from the requirement of minimal Fisher information, remains essentially unchanged in the present three-dimensional case. The reader is referred to the detailed explanations reported in I.

In I it has been shown that this principle of maximal disorder leads to an anomalous variational problem and to the following conditions for our unknown function  $L_0$ :

$$\tilde{L}(x, t) - L_0 \left( \rho, \frac{\partial \rho}{\partial x}, \frac{\partial^2 \rho}{\partial x \partial x} \right) = 0 \quad (51)$$

$$\delta \int d^3x \rho \left[ \tilde{L}(x, t) - L_0 \left( \rho, \frac{\partial \rho}{\partial x}, \frac{\partial^2 \rho}{\partial x \partial x} \right) \right] = 0, \quad (52)$$

where  $L_0$  contains only derivatives of  $\rho$  up to second order and does not explicitly depend on  $x, t$ . If Eq. (51) is taken into account, the Euler-Lagrange equations of the variational problem (52) lead to the following differential equation

$$-\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \beta}{\partial \left( \frac{\partial^2 \rho}{\partial x_k \partial x_i} \right)} + \frac{\partial}{\partial x_k} \frac{\partial \beta}{\partial \left( \frac{\partial \rho}{\partial x_k} \right)} - \frac{\partial \beta}{\partial \rho} + \frac{\beta}{\rho} = 0 \quad (53)$$

for the variable  $\beta = \rho L_0$ . Eq. (53) is a straightforward generalization of the corresponding one-dimensional relation [equation (68) of I] to three spatial dimensions.

Besides (53) a further (consistency) condition exists, which leads to a simplification of the problem. The function  $L_0$  may depend on second order derivatives of  $\rho$  but this dependence must be of a special form not leading to any terms in the Euler-Lagrange equations [according to (51) our final differential

equation for  $S$  and  $\rho$  must not contain higher than second order derivatives of  $\rho$ ]. Consequently, the first term in Eq. (53) (as well as the sum of the remaining terms) has to vanish separately and (53) can be replaced by the two equations

$$\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \beta}{\partial \left( \frac{\partial^2 \rho}{\partial x_k \partial x_i} \right)} = 0 \quad (54)$$

$$\frac{\partial}{\partial x_k} \frac{\partial \beta}{\partial \left( \frac{\partial \rho}{\partial x_k} \right)} - \frac{\partial \beta}{\partial \rho} + \frac{\beta}{\rho} = 0. \quad (55)$$

In I a new derivation of Fisher's functional has been obtained, using the general solution of the one-dimensional version of (53), as well as the so-called composition law. In the present three-dimensional situation we set ourselves a less ambitious aim. We know that Fisher's functional describes the maximal amount of disorder. If we are able to find a solution of (54), (55) that agrees with this functional (besides 'null-terms' giving no contribution to the Euler-Lagrange equations) then we will accept it as our correct solution. It is easy to see that this solution is given by

$$L_0 = B_0 \left[ -\frac{1}{2\rho^2} \sum_j \left( \frac{\partial \rho}{\partial x_j} \right)^2 + \frac{1}{\rho} \sum_j \frac{\partial^2 \rho}{\partial x_j^2} \right], \quad (56)$$

where  $B_0$  is an arbitrary constant. Eq. (56) presents again the three-dimensional (and isotropic) generalization of the one-dimensional result obtained in I. By means of the identity

$$\frac{\partial}{\partial x_i} \frac{\partial \sqrt{\rho}}{\partial x_i} \frac{\partial \sqrt{\rho}}{\partial x_k} = \frac{\partial \sqrt{\rho}}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \sqrt{\rho}}{\partial x_i} + \frac{1}{2} \frac{\partial}{\partial x_k} \frac{\partial \sqrt{\rho}}{\partial x_i} \frac{\partial \sqrt{\rho}}{\partial x_i}, \quad (57)$$

it is easily verified that the solution (56) obeys also condition (40). Using the decomposition (3) and renaming  $B$  according to  $B = \hbar^2/4m$ , the continuity equation (5) and the second differential equation (51) respectively, take the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} \frac{\rho}{m} \left( \frac{\partial S}{\partial x_k} - \frac{e}{c} A_k \right) = 0, \quad (58)$$

$$\frac{\partial S}{\partial t} + e\phi + \frac{1}{2m} \sum_k \left( \frac{\partial S}{\partial x_k} - \frac{e}{c} A_k \right)^2 + V - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = 0. \quad (59)$$

The function  $S$  occurring in (58), (59) is single-valued but not unique (not gauge-invariant). If now the complex-valued variable

$$\psi = \sqrt{\rho} e^{i \frac{S}{\hbar}}, \quad (60)$$

is introduced, the two equations (58), (59) may be written in compact form as real and imaginary parts of the linear differential equation

$$\left( \frac{\hbar}{i} \frac{\partial}{\partial t} + e\phi \right) \psi + \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}} - \frac{e}{c} \vec{A} \right)^2 \psi + V\psi = 0, \quad (61)$$

which completes our derivation of Schrödinger's equation in the presence of a gauge field.

Eq. (61) is in manifest gauge-invariant form. The gauge-invariant derivatives of  $\tilde{S}$  with respect to  $t$  and  $\vec{x}$  correspond to the two brackets in (61); in particular the canonical momentum  $\partial S/\partial \vec{x}$  corresponds to the momentum operator proportional to  $\partial/\partial \vec{x}$ . Very frequently, Eq. (61) is written in the form

$$-\frac{\hbar}{i} \frac{\partial}{\partial t} \psi = H\psi, \quad (62)$$

with the Hamilton operator

$$H = \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}} - \frac{e}{c} \vec{A} \right)^2 + V + e\phi, \quad (63)$$

This quantity is very useful despite the fact that it contains, if applied to  $\psi$ , only one of the two gauge-invariant combinations present in the original time-dependent Schrödinger equation (61). The operator (63) depends on the potentials  $\Phi$  and  $\vec{A}$  and is consequently a non-unique (not gauge-invariant) mathematical object. The same is true for the time-development operator  $U(H)$  which is an operator

function of  $H$  [20]. This non-uniqueness is a problem if  $U(H)$  is interpreted as a quantity ruling the time evolution of a single particle. It is no problem from the point of view of the statistical interpretation where  $H$  and  $U(H)$  are primarily convenient mathematical objects which occur in a natural way if the time-dependence of statistically relevant (uniquely defined) quantities, like expectation values and transition probabilities, is to be calculated.

A fundamental 'static' aspect of operators is their role as observables. The spectrum of eigenvalues of the Hamilton operator  $H$  represents the set of all possible results of energy measurements of a single particle (note that  $H$  is nevertheless a quantity characterizing an observable property of an *ensemble*; no single measurement result can be associated with  $H$ , only the measurement of the complete spectrum of eigenvalues). In this sense the Hamilton operator corresponds to the classical Hamilton function (and analogous relations are postulated for arbitrary classical observables). This correspondence is obvious in the canonical quantization procedure. It is less obvious but nevertheless visible in the present statistical approach as defined in section 2. The classical Hamilton function is implicitly contained in the statistical conditions (6) and (7). The latter comprise the differential structure of the canonical equations, which is itself determined by the Hamilton function. The formal details of the relation between classical observables and operators in the present approach have still to be worked out.

## 6 Spin - Basic theory

A simple idea to extend the present theory is to assume that sometimes - under certain external conditions to be identified later - a situation occurs where the behavior of our statistical ensemble of particles cannot longer be described by  $\rho, S$  alone but requires, e.g., the double number of field variables; let us denote these by  $\rho_1, S_1, \rho_2, S_2$  (we restrict ourselves here to this important special case which corresponds to spin one-half).

The relations defining this generalized theory should be formulated in such a way that the relations defining the previous theory (for  $\rho, S$ ) are obtained in the appropriate limits for  $\rho_1, S_1$  and  $\rho_2, S_2$ . One could say that we undertake an attempt to introduce a new (discrete) degree of freedom for the ensemble. At the present point of our investigation it is, of course, not at all clear whether or not this attempt will be successful. If we are able to derive a non-trivial set of differential equations - with coupling between  $\rho_1, S_1$  and  $\rho_2, S_2$  - then such a degree of freedom could exist in nature. If, on the contrary, we find that each pair of variables  $\rho_1, S_1$  and  $\rho_2, S_2$  obeys the same differential equations, then such a degree of freedom cannot exist (at least from the point of view of the present theory).

Using these guidelines, the basic equations of the generalized theory can be easily formulated. The probability density and probability current take the form  $\rho = \rho_1 + \rho_2$  and  $\vec{j} = \vec{j}_1 + \vec{j}_2$ , with  $\vec{j}_i$  ( $i = 1, 2$ ) defined in terms of  $\rho_i, S_i$  exactly as before (see I). Then, the continuity equation is given by

$$\frac{\partial(\rho_1 + \rho_2)}{\partial t} + \frac{\partial}{\partial x_l} \left( \frac{\rho_1}{m} \frac{\partial \tilde{S}_1}{\partial x_l} + \frac{\rho_2}{m} \frac{\partial \tilde{S}_2}{\partial x_l} \right) = 0, \quad (64)$$

where we took the possibility of multi-valuedness of the "phases" already into account, as indicated by the notation  $\tilde{S}_i$ . The statistical conditions are given by the two relations

$$\frac{d}{dt} \overline{x_k} = \frac{\overline{p_k}}{m} \quad (65)$$

$$\frac{d}{dt} \overline{p_k} = \overline{F_k^{(T)}(x, p, t)}, \quad (66)$$

which are similar to the relations used previously (in section 2 and in I), and by an additional equation

$$\frac{d}{dt} \overline{s_k} = \overline{F_k^{(R)}(x, p, t)}, \quad (67)$$

which is required as a consequence of our larger number of dynamic variables. Eq. (67) is best explained later; it is written down here for completeness. The forces  $F_k^{(T)}(x, p, t)$  and  $F_k^{(R)}(x, p, t)$  on the r.h.s. of (66) and (67) are again subject to the "statistical constraint", which has been defined in section 4. The expectation values are defined as in (8)-(10).

Performing mathematical manipulations similar to the ones reported in section 3, the l.h.s. of Eq. (66)

takes the form

$$\begin{aligned} \frac{d}{dt} \overline{p_k} = \int d^3x & \left[ \frac{\partial \rho_1}{\partial t} \frac{\partial \tilde{S}_1}{\partial x_k} + \frac{\partial \rho_2}{\partial t} \frac{\partial \tilde{S}_2}{\partial x_k} \right. \\ & \left. - \frac{\partial \rho_1}{\partial x_k} \frac{\partial \tilde{S}_1}{\partial t} - \frac{\partial \rho_2}{\partial x_k} \frac{\partial \tilde{S}_2}{\partial t} + \rho_1 \tilde{S}_{[j,k]}^{(1)} + \rho_2 \tilde{S}_{[j,k]}^{(2)} \right], \end{aligned} \quad (68)$$

where the quantities  $\tilde{S}_{[j,k]}^{(i)}$ ,  $i = 1, 2$  are defined as above [see Eq. (16)] but with  $\tilde{S}$  replaced by  $\tilde{S}_i$ .

Let us write now  $\tilde{S}$  in analogy to section 2 in the form  $\tilde{S}_i = S_i + \tilde{N}_i$ , as a sum of a single-valued part  $S_i$  and a multi-valued part  $\tilde{N}_i$ . If  $\tilde{N}_1$  and  $\tilde{N}_2$  are to represent an external influence, they must be identical and a single multi-valued part  $\tilde{N} = \tilde{N}_1 = \tilde{N}_2$  may be used instead. The derivatives of  $\tilde{N}$  with respect to  $t$  and  $x_k$  must be single-valued and we may write

$$\frac{\partial \tilde{S}_i}{\partial t} = \frac{\partial S_i}{\partial t} + e\Phi, \quad \frac{\partial \tilde{S}_i}{\partial x_k} = \frac{\partial S_i}{\partial x_k} - \frac{e}{c} A_k, \quad (69)$$

using the same familiar electrodynamic notation as in section 2. In this way we arrive at eight single-valued functions to describe the external conditions and the dynamical state of our system, namely  $\Phi$ ,  $A_k$  and  $\rho_i$ ,  $S_i$ .

In a next step we replace  $\rho_i$ ,  $S_i$  by new dynamic variables  $\rho$ ,  $S$ ,  $\vartheta$ ,  $\varphi$  defined by

$$\begin{aligned} \rho_1 &= \rho \cos^2 \frac{\vartheta}{2}, & S_1 &= S + \frac{\hbar}{2} \varphi, \\ \rho_2 &= \rho \sin^2 \frac{\vartheta}{2}, & S_2 &= S - \frac{\hbar}{2} \varphi. \end{aligned} \quad (70)$$

A transformation similar to Eq. (70) has been introduced by Takabayasi [34] in his reformulation of Pauli's equation. Obviously, the variables  $S$ ,  $\rho$  describe 'center of mass' properties (which are common to both states 1 and 2) while  $\vartheta$ ,  $\varphi$  describe relative (internal) properties of the system.

The dynamical variables  $S$ ,  $\rho$  and  $\vartheta$ ,  $\varphi$  are of course not decoupled from each other. It turns out (see below) that the influence of  $\vartheta$ ,  $\varphi$  on  $S$ ,  $\rho$  can be described in a (formally) similar way as the influence of an external electromagnetic field if a 'vector potential'  $\vec{A}^{(s)}$  and a 'scalar potential'  $\phi^{(s)}$ , defined by

$$A_l^{(s)} = -\frac{\hbar c}{2e} \cos \vartheta \frac{\partial \varphi}{\partial x_l}, \quad \phi^{(s)} = \frac{\hbar}{2e} \cos \vartheta \frac{\partial \varphi}{\partial t}, \quad (71)$$

are introduced. Denoting these fields as 'potentials', we should bear in mind that they are not externally controlled but defined in terms of the internal dynamical variables. Using the abbreviations

$$\hat{A}_l = A_l + A_l^{(s)}, \quad \hat{\phi} = \phi + \phi^{(s)}, \quad (72)$$

the second statistical condition (66) can be written in the following compact form

$$\begin{aligned} & - \int d^3x \frac{\partial \rho}{\partial x_l} \left[ \left( \frac{\partial S}{\partial t} + e\hat{\phi} \right) + \frac{1}{2m} \sum_j \left( \frac{\partial S}{\partial x_j} - \frac{e}{c} \hat{A}_j \right)^2 \right] \\ & + \int d^3x \rho \left[ -\frac{e}{c} v_j \left( \frac{\partial \hat{A}_l}{\partial x_j} - \frac{\partial \hat{A}_j}{\partial x_l} \right) - \frac{e}{c} \frac{\partial \hat{A}_l}{\partial t} - e \frac{\partial \hat{\phi}}{\partial x_l} \right] \\ & = \overline{F_l^{(T)}(x, p, t)} = \int d^3x \rho F_l^{(T)}(x, p, t), \end{aligned} \quad (73)$$

which shows a formal similarity to the spin-less case [see (23) and (32)]. The components of the velocity field in (73) are given by

$$v_j = \frac{1}{m} \left( \frac{\partial S}{\partial x_j} - \frac{e}{c} \hat{A}_j \right). \quad (74)$$

If now fields  $E_l$ ,  $B_l$  and  $E_l^{(s)}$ ,  $B_l^{(s)}$  are introduced by relations analogous to (30), the second line of (73) may be written in the form

$$\int d^3x \rho \left[ (e\vec{E} + \frac{e}{c} \vec{v} \times \vec{B})_l + (eE^{(s)} + \frac{e}{c} \vec{v} \times B^{(s)})_l \right], \quad (75)$$

which shows that both types of fields, the external fields as well as the internal fields due to  $\vartheta$ ,  $\varphi$ , enter the theory in the same way, namely in the form of a Lorentz force. In this context we note that Pauli's equation has recently been derived in the framework of a gauge theory [10].

The first, externally controlled Lorentz force in (75) may be eliminated in exactly the same manner as in section 4 by writing

$$\overline{F_l^{(T)}}(x, p, t) = \int d^3x \rho (e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B})_l + \int d^3x \rho F_l^{(I)}(x, p, t). \quad (76)$$

This means that one of the forces acting on the system as a whole is again given by a Lorentz force; there may be other nontrivial forces  $F^{(I)}$  which are still to be determined. The second 'internal' Lorentz force in (75) can, of course, not be eliminated in this way. In order to proceed, the third statistical condition (67) must be implemented. To do that it is useful to write Eq. (73) in the form

$$\begin{aligned} & - \int d^3x \frac{\partial \rho}{\partial x_l} \left[ \left( \frac{\partial S}{\partial t} + e\hat{\phi} \right) + \frac{1}{2m} \sum_j \left( \frac{\partial S}{\partial x_j} - \frac{e}{c} \hat{A}_j \right)^2 \right] \\ & + \int d^3x \frac{\hbar}{2} \rho \sin \vartheta \left( \frac{\partial \vartheta}{\partial x_l} \left[ \frac{\partial \varphi}{\partial t} + v_j \frac{\partial \varphi}{\partial x_j} \right] - \frac{\partial \varphi}{\partial x_l} \left[ \frac{\partial \vartheta}{\partial t} + v_j \frac{\partial \vartheta}{\partial x_j} \right] \right) \\ & = \overline{F_l^{(I)}}(x, p, t) = \int d^3x \rho F_l^{(I)}(x, p, t), \end{aligned} \quad (77)$$

using (75), (76) and the definition (71) of the fields  $A_l^{(s)}$  and  $\phi^{(s)}$ .

We interpret the fields  $\varphi$  and  $\vartheta$  as angles (with  $\varphi$  measured from the  $y$ -axis of our coordinate system) determining the direction of a vector

$$\vec{s} = \frac{\hbar}{2} (\sin \vartheta \sin \varphi \vec{e}_x + \sin \vartheta \cos \varphi \vec{e}_y + \cos \vartheta \vec{e}_z), \quad (78)$$

of constant length  $\frac{\hbar}{2}$ . As a consequence,  $\dot{\vec{s}}$  and  $\vec{s}$  are perpendicular to each other and the classical force  $\vec{F}^{(R)}$  in Eq. (67) should be of the form  $\vec{D} \times \vec{s}$ , where  $\vec{D}$  is an unknown field. In contrast to the 'external force', we are unable to determine the complete form of this 'internal' force from the statistical constraint [a further comment on this point will be given in section 9] and set

$$\vec{F}^{(R)} = -\frac{e}{mc} \vec{B} \times \vec{s}, \quad (79)$$

where  $\vec{B}$  is the external 'magnetic field', as defined by Eq. (30), and the factor in front of  $\vec{B}$  has been chosen to yield the correct  $g$ -factor of the electron.

The differential equation

$$\frac{d}{dt} \vec{s} = -\frac{e}{mc} \vec{B} \times \vec{s} \quad (80)$$

for particle variables  $\vartheta(t)$ ,  $\varphi(t)$  describes the rotational state of a classical magnetic dipole in a magnetic field [31]. Recall that we do *not* require that such an equation is fulfilled in the present theory. The present variables are the fields  $\vartheta(x, t)$ ,  $\varphi(x, t)$  which may be thought of as describing a kind of 'rotational state' of the statistical ensemble as a whole, and have to fulfill the averaged version (67) of (80).

Performing steps similar to the ones described in I and section 3, the third statistical condition (67) implies the following differential relations,

$$\begin{aligned} \dot{\varphi} + v_j \frac{\partial \varphi}{\partial x_j} &= \frac{e}{mc} \frac{1}{\sin \vartheta} (B_z \sin \vartheta - B_y \cos \vartheta \cos \varphi - B_x \cos \vartheta \sin \varphi) \\ &+ \frac{\cos \varphi}{\sin \vartheta} G_1 - \frac{\sin \varphi}{\sin \vartheta} G_2, \end{aligned} \quad (81)$$

$$\dot{\vartheta} + v_j \frac{\partial \vartheta}{\partial x_j} = \frac{e}{mc} (B_x \cos \varphi - B_y \sin \varphi) - \frac{G_3}{\sin \vartheta}, \quad (82)$$

for the dynamic variables  $\vartheta$  and  $\varphi$ . These equations contain three fields  $G_i(x, t)$ ,  $i = 1, 2, 3$  which have to obey the conditions

$$\int d^3x \rho G_i = 0, \quad \vec{G} \vec{s} = 0, \quad (83)$$

and are otherwise arbitrary. The 'total derivatives' of  $\varphi$  and  $\vartheta$  in (77) may now be eliminated with the help of (81),(82) and the second line of Eq. (77) takes the form

$$\begin{aligned} & \int d^3x \frac{\partial \rho}{\partial x_l} \frac{e}{mc} s_j B_j + \int d^3x \rho \frac{e}{mc} s_j \frac{\partial}{\partial x_l} B_j \\ & + \int d^3x \rho \frac{\hbar}{2} \left( \cos \varphi \frac{\partial \vartheta}{\partial x_l} G_1 - \sin \varphi \frac{\partial \vartheta}{\partial x_l} G_2 + \frac{\partial \varphi}{\partial x_l} G_3 \right). \end{aligned} \quad (84)$$

The second term in (84) presents an external macroscopic force. It may be eliminated from (77) by writing

$$\overline{F_l^{(I)}}(x, p, t) = \int d^3x \rho \left( -\mu_j \frac{\partial}{\partial x_l} B_j \right) + \overline{F_l^{(V)}}(x, p, t), \quad (85)$$

where the magnetic moment of the electron  $\mu_i = -(e/mc)s_i$  has been introduced. The first term on the r.h.s. of (85) is the expectation value of the well-known electro-dynamical force exerted by an inhomogeneous magnetic field on the translational motion of a magnetic dipole; this classical force plays an important role in the standard interpretation of the quantum-mechanical Stern-Gerlach effect. It is satisfying that both translational forces, the Lorentz force as well as this dipole force, can be derived in the present approach. The remaining unknown force  $\vec{F}^{(V)}$  in (85) leads (in the same way as in section 4) to a mechanical potential  $V$ , which will be omitted for brevity.

The integrand of the first term in (84) is linear in the derivative of  $\rho$  with respect to  $x_l$ . It may consequently be added to the first line of (77) which has the same structure. Therefore, it represents (see below) a contribution to the generalized Hamilton-Jacobi differential equation. The third term in (84) has the mathematical structure of a force term, but does not contain any externally controlled fields. Thus, it must also represent a contribution to the generalized Hamilton-Jacobi equation. This implies that this third term can be written as

$$\int d^3x \rho \frac{\hbar}{2} \left( \cos \varphi \frac{\partial \vartheta}{\partial x_l} G_1 - \sin \varphi \frac{\partial \vartheta}{\partial x_l} G_2 + \frac{\partial \varphi}{\partial x_l} G_3 \right) = \int d^3x \frac{\partial \rho}{\partial x_l} L'_0, \quad (86)$$

where  $L'_0$  is an unknown field depending on  $G_1, G_2, G_3$ .

Collecting terms and restricting ourselves, as in section 5, to an isotropic law, the statistical condition (77) takes the form of a generalized Hamilton-Jacobi equation:

$$\tilde{L} := \left( \frac{\partial S}{\partial t} + e\hat{\phi} \right) + \frac{1}{2m} \sum_j \left( \frac{\partial S}{\partial x_j} - \frac{e}{c} \hat{A}_j \right)^2 + \mu_i B_i = L_0. \quad (87)$$

The unknown function  $L_0$  must contain  $L'_0$  but may also contain other terms, let us write  $L_0 = L'_0 + \Delta L_0$ .

## 7 Spin - Fisher information

Let us summarize at this point what has been achieved so far. We have four coupled differential equations for our dynamic field variables  $\rho, S, \vartheta, \varphi$ . The first of these is the continuity equation (64), which is given, in terms of the present variables, by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_l} \left[ \frac{\rho}{m} \left( \frac{\partial S}{\partial x_j} - \frac{e}{c} \hat{A}_j \right) \right] = 0. \quad (88)$$

The three other differential equations, the evolution equations (81), (82) and the generalized Hamilton-Jacobi equation (87), do not yet possess a definite mathematical form. They contain four unknown functions  $G_i, L_0$  which are constrained, but not determined, by (83), (86).

The simplest choice, from a formal point of view, is  $G_i = L_0 = 0$ . In this limit the present theory agrees with Schiller's field-theoretic (Hamilton-Jacobi) version [31] of the equations of motion of a classical dipole. This is a classical (statistical) theory despite the fact that it contains [see (71)] a number  $\hbar$ . But this classical theory is not realized in nature; at least not in the microscopic domain. The reason is that the simplest choice from a formal point of view is not the simplest choice from a physical point of view. The postulate of maximal simplicity (Ockham's razor) implies equal probabilities and the principle of maximal entropy in classical statistical physics. A similar principle which is able to 'explain' the nonexistence of classical physics (in the microscopic domain) is the principle of minimal Fisher information [11]. The relation between the two (classical and quantum-mechanical) principles has been discussed in detail in I.

The mathematical formulation of the principle of minimal Fisher information for the present problem requires a generalization, as compared to I, because we have now several fields with coupled time-evolution equations. As a consequence, the spatial integral (spatial average) over  $\rho(\tilde{L} - L_0)$  in the variational problem (52) should be replaced by a space-time integral, and the variation should be performed with respect to all four variables. The problem can be written in the form

$$\delta \int dt \int d^3x \rho (\tilde{L} - L_0) = 0 \quad (89)$$

$$E_a = 0, \quad a = S, \rho, \vartheta, \varphi, \quad (90)$$

where  $E_a = 0$  is a shorthand notation for the equations (88), (87), (82) (81). Eqs. (89), (90) require that the four Euler-Lagrange equations of the variational problem (89) agree with the differential equations (90). This imposes conditions for the unknown functions  $L_0, G_i$ . If the *solutions* of (89), (90) for  $L_0, G_i$  are inserted in the variational problem (89), the four relations (90) become redundant and  $\rho(\tilde{L} - L_0)$  becomes the Lagrangian density of our problem. Thus, Eqs. (89) and (90) represent a method to construct a Lagrangian.

We assume a functional form  $L_0(\chi_\alpha, \partial_k \chi_\alpha, \partial_k \partial_l \chi_\alpha)$ , where  $\chi_\alpha = \rho, \vartheta, \varphi$ . This means  $L_0$  does not possess an explicit  $x, t$ -dependence and does not depend on  $S$  (this would lead to a modification of the continuity equation). We further assume that  $L_0$  does not depend on time-derivatives of  $\chi_\alpha$  (the basic structure of the time-evolution equations should not be affected) and on spatial derivatives higher than second order. These second order derivatives must be taken into account but should not give contributions to the variational equations (a more detailed discussion of the last point has been given in I).

The variation with respect to  $S$  reproduces the continuity equation which is unimportant for the determination of  $L_0, G_i$ . Performing the variation with respect to  $\rho, \vartheta, \varphi$  and taking the corresponding conditions (87), (82) (81) into account leads to the following differential equations for  $L_0, G_1 \cos \varphi - G_2 \sin \varphi$  and  $G_3$ ,

$$-\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \rho L_0}{\partial \frac{\partial^2 \rho}{\partial x_k \partial x_i}} + \frac{\partial}{\partial x_k} \frac{\partial \rho L_0}{\partial \frac{\partial \rho}{\partial x_k}} - \rho \frac{\partial L_0}{\partial \rho} = 0 \quad (91)$$

$$-\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \rho L_0}{\partial \frac{\partial^2 \vartheta}{\partial x_k \partial x_i}} + \frac{\partial}{\partial x_k} \frac{\partial \rho L_0}{\partial \frac{\partial \vartheta}{\partial x_k}} - \frac{\partial \rho L_0}{\partial \vartheta} - \frac{\hbar \rho}{2} (G_1 \cos \varphi - G_2 \sin \varphi) = 0 \quad (92)$$

$$-\frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \frac{\partial \rho L_0}{\partial \frac{\partial^2 \varphi}{\partial x_k \partial x_i}} + \frac{\partial}{\partial x_k} \frac{\partial \rho L_0}{\partial \frac{\partial \varphi}{\partial x_k}} - \frac{\partial \rho L_0}{\partial \varphi} - \frac{\hbar}{2} \rho G_3 = 0. \quad (93)$$

The variable  $S$  does not occur in (91)-(93) in agreement with our assumptions about the form of  $L_0$ . It is easy to see that a proper solution (with vanishing variational contributions from the second order derivatives) of (91)-(93) is given by

$$L_0 = \frac{\hbar^2}{2m} \left[ \frac{1}{\sqrt{\rho}} \frac{\partial}{\partial \vec{x}} \frac{\partial}{\partial \vec{x}} \sqrt{\rho} - \frac{1}{4} \sin^2 \vartheta \left( \frac{\partial \varphi}{\partial \vec{x}} \right)^2 - \frac{1}{4} \left( \frac{\partial \vartheta}{\partial \vec{x}} \right)^2 \right] \quad (94)$$

$$\hbar G_1 \cos \varphi - \hbar G_2 \sin \varphi = \frac{\hbar^2}{2m} \left[ \frac{1}{2} \sin 2\vartheta \left( \frac{\partial \varphi}{\partial \vec{x}} \right)^2 - \frac{1}{\rho} \frac{\partial}{\partial \vec{x}} \rho \frac{\partial \vartheta}{\partial \vec{x}} \right] \quad (95)$$

$$\hbar G_3 = -\frac{\hbar^2}{2m} \frac{1}{\rho} \frac{\partial}{\partial \vec{x}} \left( \rho \sin^2 \vartheta \frac{\partial \varphi}{\partial \vec{x}} \right). \quad (96)$$

A new adjustable parameter appears on the r.h.s of (94)- (96) which has been identified with  $\hbar^2/2m$ , where  $\hbar$  is again Planck's constant. This second  $\hbar$  is related to the quantum-mechanical principle of maximal disorder. It is in the present approach not related in any obvious way to the previous "classical"  $\hbar$  which denotes the amplitude of a rotation.

The solutions for  $G_1, G_2$  may be obtained with the help of the second condition ( $\vec{G}\vec{s} = 0$ ) listed in Eq. (83). The result may be written in the form

$$G_1 = \frac{\hbar}{2m} \frac{1}{\rho} \frac{\partial}{\partial \vec{x}} \rho \left( \frac{1}{2} \sin 2\vartheta \sin \varphi \frac{\partial \varphi}{\partial \vec{x}} - \cos \varphi \frac{\partial \vartheta}{\partial \vec{x}} \right) \quad (97)$$

$$G_2 = \frac{\hbar}{2m} \frac{1}{\rho} \frac{\partial}{\partial \vec{x}} \rho \left( \frac{1}{2} \sin 2\vartheta \cos \varphi \frac{\partial \varphi}{\partial \vec{x}} + \sin \varphi \frac{\partial \vartheta}{\partial \vec{x}} \right).$$

Eqs. (96) and (97) show that the first condition listed in (83) is also satisfied. The last condition is also fulfilled:  $L_0$  can be written as  $L'_0 + \Delta L_0$ , where

$$L'_0 = -\frac{\hbar^2}{8m} \left[ \sin^2 \vartheta \left( \frac{\partial \varphi}{\partial \vec{x}} \right)^2 - \left( \frac{\partial \vartheta}{\partial \vec{x}} \right)^2 \right], \quad \Delta L_0 = \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial}{\partial \vec{x}} \frac{\partial}{\partial \vec{x}} \sqrt{\rho}, \quad (98)$$

and  $L'_0$  fulfills (86). We see that  $L'_0$  is a quantum-mechanical contribution to the rotational motion while  $\Delta L_0$  is related to the probability density of the ensemble (as could have been guessed considering the mathematical form of these terms). The last term is the same as in the spinless case [see (59)].

The remaining task is to show that the above solution for  $L_0$  does indeed lead to a (appropriately generalized) Fisher functional. This can be done in several ways. The simplest is to use the following result due to Reginatto [28]:

$$\int d^3x (-\rho L_0) = \frac{\hbar^2}{8m} \sum_{j=1}^3 \int d^3x \sum_{k=1}^3 \frac{1}{\rho^{(j)}} \left( \frac{\partial \rho^{(j)}}{\partial x_k} \right)^2, \quad (99)$$

$$\rho^{(1)} := \rho \sin^2 \frac{\vartheta}{2} \cos^2 \frac{\varphi}{2}, \quad \rho^{(2)} := \rho \sin^2 \frac{\vartheta}{2} \sin^2 \frac{\varphi}{2}, \quad \rho^{(3)} := \rho \cos^2 \frac{\vartheta}{2}. \quad (100)$$

The functions  $\rho^{(j)}$  represent the probability that a particle is at space-time point  $x, t$  and  $\vec{s}$  points into direction  $j$ . Inserting (94) the validity of (99) may easily be verified. The r.h.s. of Eq. (99) shows that the averaged value of  $L_0$  represents indeed a Fisher functional, which completes our calculation of the 'quantum terms'  $L_0, G_i$ .

Summarizing, our assumption, that under certain external conditions four state variables instead of two may be required, led to a nontrivial result, namely the four coupled differential equations (88), (87), (82) (81) with  $L_0, G_i$  given by (94), (97), (96). The external condition which stimulates this splitting is given by a gauge field; the most important case is a magnetic field  $\vec{B}$  but other possibilities do exist (see below). These four differential equations are equivalent to the much simpler differential equation

$$\left( \frac{\hbar}{i} \frac{\partial}{\partial t} + e\phi \right) \hat{\psi} + \frac{1}{2m} \left( \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}} - \frac{e}{c} \vec{A} \right)^2 \hat{\psi} + \mu_B \vec{\sigma} \vec{B} \hat{\psi} = 0, \quad (101)$$

which is linear in the complex-valued two-component state variable  $\hat{\psi}$  and is referred to as Pauli equation (the components of the vector  $\vec{\sigma}$  are the three Pauli matrices and  $\mu_B = -e\hbar/2mc$ ). To see the equivalence one writes [34],[14]

$$\hat{\psi} = \sqrt{\rho} e^{\frac{i}{\hbar} S} \begin{pmatrix} \cos \frac{\vartheta}{2} e^{i \frac{\varphi}{2}} \\ i \sin \frac{\vartheta}{2} e^{-i \frac{\varphi}{2}} \end{pmatrix}, \quad (102)$$

and evaluates the real and imaginary parts of the two scalar equations (101). This leads to the four differential equations (88), (87), (82) (81) and completes the present spin theory.

In terms of the real-valued functions  $\rho, S, \vartheta, \varphi$  the quantum-mechanical solutions (94),(96),(97) for  $L_0, G_i$  look complicated in comparison to the classical solutions  $L_0 = G_i = 0$ . In terms of the variable  $\hat{\psi}$  the situation changes to the contrary: The quantum-mechanical equation becomes simple (linear) and the classical equation, which has been derived by Schiller [31], becomes complicated (nonlinear). It is satisfying that the simplicity of the underlying physical principle (principle of maximal disorder) leads to a simple mathematical representation of the final basic equation.

Besides the Pauli equation we found, as a second important result of our spin calculation, that the following local force is compatible with the statistical constraint:

$$\vec{F}^L + \vec{F}^I = e \left( \vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right) - \vec{\mu} \cdot \frac{\partial}{\partial \vec{x}} \vec{B}. \quad (103)$$

Here, the velocity field  $\vec{v}(x, t)$  [see (34),(35)] and the magnetic moment field  $\vec{\mu}(x, t) = -(e/mc)\vec{s}(x, t)$  [see (78),(85)] have been replaced by corresponding particle quantities  $\vec{v}(t)$  and  $\vec{\mu}(t)$ ; the dot denotes the inner product between  $\vec{\mu}$  and  $\vec{B}$ . The first of the two forces in (103) is the Lorentz force; it has been derived here (basically in the same manner as in the spinless case) from first principles without any additional assumptions. The same cannot be said about the second force which takes this particular form as a consequence of some additional assumptions concerning the form of the 'internal force'  $\vec{F}^R$  [see (79)]. In particular, the field appearing in  $\vec{F}^R$  was arbitrary as well as the proportionality constant



(g-factor of the electron) in front of it. It is well-known that in a relativistic treatment the spin term appears automatically if the potentials are introduced. Interestingly, this unity is not restricted to the relativistic regime but may as well be created in non-relativistic treatments [4],[12]. A simple indication of this unity is the fact that the spin term contains no new fundamental constants. One might speculate that an analogous formulation for the present theory would automatically eliminate the above mentioned shortcoming. A most natural framework to study this point - which is a subject of future research - is probably a relativistic one.

In the present treatment spin has been introduced as a property of an ensemble and not of individual particles. Of course, the properties of an ensemble cannot be thought of as being completely independent from the properties of the particles it is made from. The question whether or not a property 'spin' can be ascribed to single particles is a subtle one. Formally, we could assign a probability of being in a state  $i$  ( $i = 1, 2$ ) to a particle just as we assign a probability for being at a position  $\vec{x} \in R^3$ . The problem is that, contrary to position, no classical meaning - and no classical measuring device - can be associated with the discrete degree of freedom  $i$ . Experimentally, the measurement of the 'spin of a single electron' is - in contrast to the measurement of its position - a notoriously difficult task. Such experiments, and a number of other interesting questions related to spin, have been discussed by Morrison [26].

## 8 The classical limit

The classical limit of Schrödinger's equation plays an important role for two topics discussed in the next section, namely the interpretation of quantum mechanics and the particular significance of potentials in quantum mechanics. This 'classical limit theory' [18] is given by the two differential equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k} \frac{\rho}{m} \left( \frac{\partial S}{\partial x_k} - \frac{e}{c} A_k \right) = 0, \quad (104)$$

$$\frac{\partial S}{\partial t} + e\phi + \frac{1}{2m} \sum_k \left( \frac{\partial S}{\partial x_k} - \frac{e}{c} A_k \right)^2 + V = 0, \quad (105)$$

which are obtained from Eqs. (58) and (59) by performing the limit  $\hbar \rightarrow 0$ . The quantum mechanical theory (58) and (59) and the classical theory (104) and (105) show fundamentally the same mathematical structure; both are initial value problems for the variables  $S$  and  $\rho$  obeying two partial differential equations. The difference is the absence of the last term on the l.h.s. of (59) in the corresponding classical equation (105). This leads to a decoupling of  $S$  and  $\rho$  in (105); the identity of the classical object described by  $S$  is no longer affected by statistical aspects described by  $\rho$ . The mathematical relations between quantum states on the one hand, and 'families' of solutions of the classical equation (105) on the other hand, have been studied by Slawianowski [33].

The field theory (104),(105) for the two 'not decoupled' fields  $S$  and  $\rho$  is obviously very different from classical mechanics which is formulated in terms of trajectories. The fact that one of these equations, namely (105), agrees with the Hamilton-Jacobi equation, does not change the situation since the presence of the continuity equation (104) cannot be neglected. On top of that, even if it could be neglected, Eq. (105) would still be totally different from classical mechanics: In order to construct particle trajectories from the partial differential equation (105) for the field  $S(x, t)$ , a number of clearly defined mathematical manipulations, which are part of the classical theory of canonical transformations [13], must be performed. The crucial point is that the latter theory is *not* part of quantum mechanics and cannot be added 'by hand' in the limit  $\hbar \rightarrow 0$ . Thus, (104),(105) is - like quantum mechanics - an indeterministic theory predicting not values of single event observables but only probabilities, which must be verified by ensemble measurements.

Given that we found a solution  $S(x, t)$ ,  $\rho(x, t)$  of (104), (105) for given initial values, we may ask which experimental predictions can be made with the help of these quantities. Using the fields  $\vec{p}(x, t)$ ,  $\vec{E}(x, t)$  defined by Eqs. (4), (3), the Hamilton-Jacobi equation (105) takes the form

$$\frac{\vec{p}^2(x, t)}{2m} + V(x, t) = \vec{E}(x, t), \quad (106)$$

The l.h.s. of (106) depends on the field  $\vec{p}$  in the same way as a classical particle Hamiltonian on the (gauge-invariant) kinetic momentum  $\vec{p}$ . We conclude that the field  $\vec{p}(x, t)$  describes a mapping from space-time points to particle momenta: If a particle (in an external electromagnetic field) is found at time  $t$  at the point  $x$ , then its kinetic momentum is given by  $\vec{p}(x, t)$ . This is not a deterministic prediction

since we can not predict if a single particle will be or will not be at time  $t$  at point  $x$ ; the present theory gives only a probability  $\rho(x, t)$  for such an event. Combining our findings about  $p$  and  $x$  we conclude that the experimental prediction which can be made with the help of  $S(x, t)$ ,  $\rho(x, t)$  is given by the following phase space probability density:

$$w(x, p, t) = \rho(x, t) \delta^{(3)}\left(p - \frac{\partial \tilde{S}(x, t)}{\partial x}\right). \quad (107)$$

Eq. (107) confirms our claim that the classical limit theory is a statistical theory. The one-dimensional version of (107) has been obtained before by means of a slightly different method [18]. The deterministic element [realized by the delta-function shaped probability in (107)] contained in the classical statistical theory (104), (105) is *absent* in quantum mechanics [18].

Eqs. (104), (105) constitute the mathematically well-defined limit  $\hbar \rightarrow 0$  of Schrödinger's equation. Insofar as there is general agreement with regard to two points, namely that (i) 'non-classicality' (whatever this may mean precisely) is expressed by a nonzero  $\hbar$ , and that (ii) Schrödinger's equation is the most important relation of quantum theory, one would also expect general agreement with regard to a further point, namely that Eqs. (104), (105) present essentially (for a three-dimensional configuration space) the *classical limit* of quantum mechanics. But this is, strangely enough, not the case. With a few exceptions [35],[30], [6],[32],[19] most works (too many to be quoted) take it for granted that the classical limit of quantum theory is classical mechanics. The objective of papers [29],[36],[22],[2] devoted to ".the classical limit of quantum mechanics.." is very often not the problem: "*what is* the classical limit of quantum mechanics?" but rather: "*how to bridge* the gap between quantum mechanics and classical mechanics?". Thus, the fact that classical mechanics is the classical limit of quantum mechanics is considered as *evident* and any facts not compatible with it - like Eqs. (104), (105) - are denied.

What, then, is the reason for this widespread denial of reality? The reason is the principle of reductionism which still rules the thinking of most physicists today. The reductionistic ideal is a hierarchy of physical theories; better theories have an enlarged domain of validity and contain 'inferior' theories as special cases. This principle which has been extremely successful in the past *dictates* that classical mechanics is a special case of quantum theory. Successful as this idea might have been during a long period of time it is not necessarily universally true; quantum mechanics and classical mechanics describe different domains of reality, both may be true in their own domains of validity. Many phenomena in nature indicate that the principle of reductionism (alone) is insufficient to describe reality [23]. Releasing ourselves from the metaphysical principle of reductionism, we accept that the classical limit of quantum mechanics for a three-dimensional configuration space is not classical mechanics but the statistical theory defined by Eqs. (104), (105). It is clear that this theory is not realized in nature (with the same physical meaning of the variables) because  $\hbar$  is different from zero. But this is a different question and does not affect the conclusion.

## 9 Discussion

In section 4 it has been shown that only the Lorentz force can exist as fundamental (quantum mechanical) force if the statistical assumptions of section 3 are true. It is the only force (see however the remarks on spin below) that can be incorporated in a 'standard' differential equation for the dynamical variables  $\rho$ ,  $S$ . The corresponding terms in the statistical field equations, *representing* the Lorentz force, must be given by the familiar gauge (minimal) coupling terms containing the potentials. The important fact that all forces in nature follow this 'principle of minimal coupling' is commonly explained as a consequence of local gauge invariance. The present treatment offers an alternative statistical explanation.

Let us use the following symbolic notation to represent the relation between the local force and the terms representing its action in a statistical context:

$$\Phi, \vec{A} \Rightarrow e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}. \quad (108)$$

The fields  $\vec{E}$  and  $\vec{B}$  are uniquely defined in terms of the potentials  $\phi$  and  $\vec{A}$  [see (30)] while the inverse is not true. Roughly speaking, the local fields are 'derivatives' of the potentials - and the potentials are 'integrals' of the local field; this mathematical relation reflects the physical role of the potentials  $\phi$  and  $\vec{A}$  as statistical representatives of the local fields  $\vec{E}$  and  $\vec{B}$ , as well as their non-uniqueness. It might seem that the logical chain displayed in (108) is already realized in the classical treatment of a particle-field system, where potentials have to be introduced in order to construct a Lagrangian [21].

However, in this case, the form of the local force is not derived but postulated. The present treatment 'explains' the form of the Lagrangian - as a consequence of the basic assumptions listed in section 3.

The generalization of the present theory to spin, reported in sections 6 and 7, leads to a correspondence similar to Eq. (108), namely

$$\vec{\mu}\vec{B} \rightarrow \vec{\mu} \cdot \frac{\partial}{\partial \vec{x}} \vec{B}. \quad (109)$$

The term linear in  $\vec{B}$ , on the l.h.s. of (109), plays the role of a 'potential' for the local force on the r.h.s. All points discussed after Eq. (108) apply here as well [As a matter of fact we consider  $\vec{B}$  as a unique physical quantity; it would not be unique if it would be defined in terms of the tensor on the r.h.s. of (109)]. We see that the present approach allows for a certain unification of the usual gauge and spin interaction terms - comparable with a relativistic formulation, where the spin coupling is introduced simultaneously with the other gauge coupling terms. Unfortunately, the derivation of the spin force on the r.h.s. of (109) differs from the derivation of the Lorentz force insofar as additional assumptions had to be made in order to arrive at the final result (see the remarks in section 7).

Our notation for potentials  $\phi$ ,  $\vec{A}$ , fields  $\vec{E}$ ,  $\vec{B}$ , and parameters  $e$ ,  $c$  suggests that these quantities are electro-dynamical in nature. However, this is not necessarily true. The constraint (31) yields four equations which are not sufficient to determine the six fields  $\vec{E}$ ,  $\vec{B}$ ; additional conditions must be imposed. The most familiar possibility is, of course, the second pair of Maxwell's equations leading to the electro-dynamical gauge field. A second possible realization for the fields  $\vec{E}$ ,  $\vec{B}$  is given by the *inertial* forces acting on a mass  $m$  in an arbitrarily accelerated reference frame [15]. In Appendix A a brief discussion of the inertial gauge field and its interplay with the Maxwell field is given. It is remarkable that the present theory establishes a (admittedly somewhat vague) link between the two extremely separated physical fields of inertia and quantum theory. An interesting point is that the mathematical axioms of  $U(1)$  gauge theory imply the Maxwell field but say nothing about the inertial field which is just as real as the former. The inertial field  $\vec{B}_I$  [see (114)] may also lead to a spin response of the ensemble. Experiments to verify such inertial effects of spin have been proposed by Mashhoon and Kaiser [25].

It is generally assumed that the electro-dynamic potentials have a particular significance in quantum mechanics which they do not have in classical physics. Let us analyze this statement in detail. (we restrict ourselves in the following discussion to the electro-dynamical gauge field). The first part of this statement, concerning the significance of the potentials, is of course true. The second part of the statement, asserting that in classical physics all external influences *can* be described solely in terms of field strengths, is wrong. More precisely, it is true for classical mechanics but not for classical physics in general. A counterexample - a theory belonging to classical physics but with potentials playing an indispensable role - is provided by the classical limit (104),(105) of Schrödinger's equation. In this field theory the potentials play an indispensable role because (in contrast to particle theories, like the canonical equations) no further derivatives of the Hamiltonian, which could restore the fields, are to be performed. This means that the significance of the potentials is not restricted to quantum theory but rather holds for the whole class of *statistical* theories discussed above, which contains both quantum theory and its classical limit theory as special cases. This result is in agreement with the present interpretation of the potentials as statistical representatives of the local fields of particle physics.

The precise characterization of the role of the potentials is important for the interpretation of the Aharonov-Bohm effect. The 'typical quantum-mechanical features' observed in these phase shift experiments should be identified by comparing the quantum mechanical results not with classical mechanics but with the predictions of the classical statistical theory (104),(105). The predictions of two statistical theories, both of which use potentials to describe the influence of the external field, have to be compared.

The interpretation of a physical theory is commonly considered as a matter that cannot be described by mathematical means. This may be generally true, but nevertheless mathematical facts exist which have an immediate bearing on questions of interpretation. The most important of these is probably the limiting behavior of a theory. The fact that the classical limit of quantum mechanics - discussed in section 8 - is not classical mechanics but a special (configuration space) classical statistical theory is widely unknown. This lack of knowledge is one of the main reasons for the widespread erroneous belief that quantum mechanics can be used to describe the dynamics of individual particles. Unfortunately, this erroneous belief is historically grown and firmly established in our thinking as shown by the ubiquitous use of phrases like 'the wave function of the electron' or 'quantum *mechanics*'.

It is clear that an erroneous identification of the domain of validity of a physical theory will automatically create all kinds of mysteries, ill-posed questions and unsolvable problems. Above, we have identified one of the more subtle problems of this kind, concerning the role of potentials in quantum mechanics (a paradigmatic example is the 'measurement problem' which is unsolved since its creation eighty years

ago). Generalizing the above argumentation concerning potentials, we claim that characteristic features of quantum theory cannot be identified by comparison with classical mechanics. Instead, quantum theory should be compared with its classical limit, which is in the present 3D-case given by (104), (105). The latter theory is probably much more difficult to solve (numerically) than Schrödinger's equation because it is no longer linear in  $\psi$ . Nevertheless, it would be very interesting to compare the solutions of (104), (105) with those of (58), (59) to find out which 'typical quantum-theoretic features' are already given by statistical (nonlocal) correlations of the classical limit theory and which features are really quantum-theoretical in nature - related to the nonzero value of  $\hbar$ .

## 10 Concluding remarks

In I it has been shown that Schrödinger's equation, which represents already an essential part of the quantum-theoretical formalism, can be derived from a number of statistical assumptions. In the present paper this theory has been generalized to gauge fields and spin. The treatment of gauge fields led to several remarkable new insights: We understand now why potentials (and not local fields) occur in the field equations of quantum theory. The non-uniqueness of the potentials and the related concept of gauge invariance becomes a comprehensible matter and is not a mystery any more. The functional form of the corresponding local force (Lorentz force) can also be derived. Both the results of I and of the present paper should, in our opinion, be interpreted as arguments in favor of the statistical interpretation of quantum theory. The remaining crucial question, in the non-relativistic domain of physics, is whether the present approach can be generalized to many particles. Assuming that this can be done, a further major problem for future research is a relativistic formulation of the present theory.

## A Electrodynamical and inertial gauge fields

A possible set of additional assumptions selecting Maxwell's theory from the class of all possible gauge fields is given by

$$\Phi(x, t) = \int d^3x' \frac{\varrho(x', t')}{|\vec{x} - \vec{x}'|}, \quad A_k(x, t) = \frac{1}{c} \int d^3x' \frac{\mathcal{J}_k(x', t')}{|\vec{x} - \vec{x}'|}, \quad (110)$$

Here, the potentials are written in terms of four localized functions  $\varrho$  and  $\mathcal{J}_k$ , where  $t' = t - |\vec{x} - \vec{x}'|/c$  and  $\varrho$  and  $\mathcal{J}_k$  obey a continuity equation. The quantities (110) are well known as retarded potentials in Lorentz gauge [16]. Using the representation (110) it may easily be shown that the corresponding fields  $\vec{E}$  and  $\vec{B}$  obey the full set of Maxwell's equations, including the two inhomogeneous equations with sources  $\varrho$  and  $\vec{\mathcal{J}}$ ,

$$\frac{\partial \vec{E}}{\partial \vec{r}} = 4\pi\varrho, \quad \frac{\partial}{\partial \vec{r}} \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{\mathcal{J}}. \quad (111)$$

An alternative possibility is, of course, to postulate the validity of Eqs. (111) for the fields directly, without making use of potentials. The interesting problem of the coexistence of the Galilei-invariant Schrödinger equation and the Lorentz-invariant Maxwell equations has been discussed recently [9].

It is less well-known that inertial forces and linearized gravitational forces provide an alternative realization, besides electrodynamics, for the fields  $\vec{E}$ ,  $\vec{B}$  [15]. We restrict ourselves here to a discussion of inertial forces, whose description in a quantum-theoretical context seems particularly interesting. Let us consider a particle in an arbitrary accelerated reference frame, whose movement relative to an inertial frame is given by a translation vector  $\vec{x}_0(t)$  and a rotation velocity  $\vec{\omega}(t)$ . As is well known such a particle experiences an inertial force field

$$\vec{F}_I(x, \dot{x}, t) = -m\ddot{\vec{a}}(t) - 2m\vec{\omega}(t) \times \dot{\vec{x}} - m\vec{\omega}(t) \times (\vec{\omega}(t) \times \vec{x}) - m\dot{\vec{\omega}}(t) \times \vec{x}, \quad (112)$$

where  $\ddot{\vec{a}}(t) = \ddot{\vec{x}}_0(t)$ . In order to use Eq. (112) in the present field theory, the particle velocity  $\dot{\vec{x}}$  has to be replaced by the velocity field (34) and the electric charge  $e$  by the 'gravitational charge'  $m$ . Then, comparison of Eqs. (112) and (35) shows that inertial forces are described by a field

$$\vec{F}_I(x, t) = m\vec{E}_I(x, t) + \frac{m}{c} \vec{\omega}(t) \times \vec{B}_I(x, t), \quad (113)$$

where

$$\vec{B}_I(x, t) = 2c\vec{\omega}(t) \quad (114)$$

$$\vec{E}_I(x, t) = -\ddot{\vec{a}}(t) - \vec{\omega}(t) \times [\vec{\omega}(t) \times \vec{x}] - \dot{\vec{\omega}}(t) \times \vec{x}. \quad (115)$$

Both inertial fields have the dimension of an acceleration. It is easy to see that they obey the homogeneous Maxwell equations (31) and that appropriate potentials, as defined by (30), are given by

$$\phi_I(x, t) = \vec{x}\vec{a}(t) - \frac{1}{2} [\vec{x} \times \vec{\omega}(t)]^2 \quad (116)$$

$$\vec{A}_I(x, t) = c\vec{\omega}(t) \times \vec{x}. \quad (117)$$

We see that inertial forces do also fit into the above gauge scheme although fundamental differences exist. In contrast to the electrodynamic field there are no additional field equations for the inertial field. Rather, its space-time dependence is more or less 'rigid' and can only be influenced by means of the input parameters  $\vec{x}_0(t)$  and  $\vec{\omega}(t)$ . Inertial fields do in contrast to electrodynamic fields not fit into the mathematical scheme gauge theory.

The fact that electrodynamic and inertial (gravitational) fields share a common (gauge) constraint, is sometimes interpreted as an indication of a common origin of both theories. We do not want to discuss this fascinating hypothesis here but mention only that these two gauge fields may also occur simultaneously; such a situation may simply be described by means of a linear combination of fields and potentials. Of course, the electrodynamic fields  $\vec{E}$  and  $\vec{B}$  are now defined with respect to the accelerated coordinate frame. (The homogeneous Maxwell equations hold in the accelerated frame as well; this is a condition for the potentials to exist. On the other hand the two inhomogeneous Maxwell equations change their form in accelerated coordinate frames [8]). In order to obtain Schrödinger's equation for a statistical ensemble of charged particles in arbitrary accelerated reference frames, the replacements

$$\frac{e}{c}\vec{A} \longrightarrow \frac{e}{c}\vec{A} + \frac{m}{c}\vec{A}_I \quad (118)$$

$$e\phi \longrightarrow e\phi + m\phi_I \quad (119)$$

have to be performed in Eq. (61). The resulting theory is invariant with respect to the gauge transformation

$$S' = S + \chi \quad (120)$$

$$\phi' = \phi + \frac{1}{e} \frac{\partial \chi_1}{\partial t}, \quad \phi'_I = \phi_I + \frac{1}{m} \frac{\partial \chi_2}{\partial t} \quad (121)$$

$$\vec{A}' = \vec{A} - \frac{c}{e} \frac{\partial \chi_1}{\partial \vec{x}}, \quad \vec{A}'_I = \vec{A}_I - \frac{c}{m} \frac{\partial \chi_2}{\partial \vec{x}} \quad (122)$$

$$\chi = \chi_1 + \chi_2 + C. \quad (123)$$

If the gauge for the inertial potentials is fixed according to (116), (117) and the above replacements are performed, the present approach leads to a very simple derivation of the quantum-theoretical Hamiltonian for an ensemble of charged particles under the influence of an electrodynamic field in a non-inertial reference frame [8].

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